Octonion Algebra and its Connection to Physics

Richard Lockyer, February 16, 2008

A Brief Introduction

Hamilton's Quaternions and Graves' Octonions are the last two in a sequence of four increasingly complicated normed division algebras; the real numbers **R**, complex numbers **C**, the Quaternions **H**, and the Octonions **O**.

The complex numbers differ from the real numbers by the addition of a second dimension whose elements are proportional to the square root of the real number –1. Another way of stating this is to say the unit of proportionality called "imaginary", when multiplied by itself produces a result of –1 real. Real multiplied by real results in real, and real multiplied by imaginary results in imaginary. The products are all commutative.

H and **O** algebras take this notion of imaginary units a bit further. **H** algebra has one real (scalar) component and three different non-scalar (vector) units. **O** algebra has one scalar and seven different vector units. Like the imaginary unit of complex numbers, the square of each vector unit is –1 real, and the commutative scalar-vector unit products retain the characteristics of the vector component. Unlike the complex numbers, there are multiple non-real unit definitions that need multiplication rules between themselves.

Each of the vector **H** or **O** units anti-commutes with every other vector unit under the operation of multiplication, meaning the result sign changes when the order of multiplication changes. This leads to multiplicative non-commutation for the division algebras **H** and **O**. **H** algebra is associative for the operation of multiplication, while **O** algebra is generally non-associative for multiplication. In addition to the anti-commutation of unlike vector units, the product of two unlike vector units is always a third vector unit, and this triplet of units forms a closed group under multiplication. This fact allows the multiplication of unlike vector units to be simply defined by cyclic permutation triplets.

Let the intrinsic units for the algebras be u_j where j runs 0 to 3 for **H**, and from 0 to 7 for **O**. The index 0 is used in both cases for the scalar unit. For both **H** and **O**,

$u_0 * u_j = u_j = u_j * u_0$ for any j
$u_k * u_k = -u_0$ for any k not 0

For **H** algebra,

$u_r * u_s = e_{rst}\, u_t$ for r,s,t different and not 0

The permutation triplet (123) defines the unlike vector unit products as

$u_1 * u_2 = u_3$, $u_2 * u_3 = u_1$, $u_3 * u_1 = u_2$, and $u_2 * u_1 = -u_3$, $u_3 * u_2 = -u_1$, $u_1 * u_3 = -u_2$.

Techniques for Generating H and O Algebra Multiplication Tables

Two algebraic systems are isomorphic if and only if their multiplication tables are equivalent. There are two isomorphic **H** algebras. They could be called right and left handed systems in identical fashion to rule variation in the common 3D xyz vector product. For unit triplet 1, 2 and 3, call the permutation triplet (123) right handed and (132) left handed. The fact that these two algebraic representations are isomorphic can be seen by first recognizing that the numeric values we assign to the permutation positions and hence rows and columns of the multiplication table are arbitrary choices, they are simply names that do not by themselves fundamentally define the algebra. We can therefore in the right handed system rename 2 to 3 and 3 to 2, producing the left handed system without modifying the multiplication table. The chiral differences of the left and right hand **H** representations only manifest themselves after we have assigned extra-algebraic significance to the unit names themselves, like physical xyz directions that remain consistent through the rename mapping of algebraic units.

It is well known that it takes seven permutation triplets to define all product combinations of two different vector units for **O**. What is not widely known is how many forms these permutations can take.

Each **O** vector unit appears in three of the seven permutation triplets in specific juxtapositions to the other two units. As previously mentioned, the names we attach to these units are arbitrary. If one places significance on the published concepts of index cycling unit numbers [4][6], this non- **O** extra-algebraic concept narrows the choice to what works for the desired real number operations on the names themselves. My preferred arbitrary choice on naming comes out of an association between binary numbers derived from the binary weights of second order polynomials and the operation of modulo-two polynomial addition.. This is similar to the Galois Field presentation in Geoffrey Dixon's fine book [4].

Make associations between the weights of second order modulo two polynomials and unit names:

1 :: {001} 2 :: {010} 3 :: {011} 4 :: {100}
5 :: {101} 6 :: {110} 7 :: {111}

Using the addition properties of modulo two polynomials:

1+2 = {001} + {010} = {011} = 3
2+3 = {010} + {011} = {001} = 1
3+1 = {011} + {001} = {010} = 2

7+6 = {111} + {110} = {001} = 1
6+1 = {110} + {001} = {111} = 7
1+7 = {001} + {111} = {110} = 6

5+7 = {101} + {111} = {010} = 2
7+2 = {111} + {010} = {101} = 5
2+5 = {010} + {101} = {111} = 7

6+5 = {110} + {101} = {011} = 3
5+3 = {101} + {011} = {110} = 6
3+6 = {011} + {110} = {101} = 5

1+4 = {001} + {100} = {101} = 5

4+5 = {100} + {101} = {001} = 1
5+1 = {101} + {001} = {100} = 4

2+4 = {010} + {100} = {110} = 6
4+6 = {100} + {110} = {010} = 2
6+2 = {110} + {010} = {100} = 4

3+4 = {011} + {100} = {111} = 7
4+7 = {100} + {111} = {011} = 3
7+3 = {111} + {011} = {100} = 4

Examining these sums, there are 7 separate closed sets of 3 unit names where each name appears in 3 of the 7 sets. These triplet choices are therefore suitable for the Octonion permutations. So far, the permutation groupings are set, but the order of units within these permutations is not yet defined. I will first continue on the modulo-two polynomial path, and then follow up with a complete definition of every possible valid set of order.

The modulo-two products of code words and a generator polynomial can create cyclic modulo two polynomial sets. The generator is always a root of the polynomial p^n+1 for some positive integer n.

The roots of the polynomial p^7+1 are:
{10000001} = {11} * {1101} * {1011}

This factorization will yield two generator polynomials:

Generator 1 = {11} * {1011} = {11101}
Generator 2 = {11} * {1101} = {10111}

Since the distributive property of multiplication holds for modulo-two polynomial multiplication, the three cyclic codes created by multiplying either of these generators by each of the polynomials representing one of our permutations will also form a closed set under the operation of modulo-two addition. For instance:

{111} * {10111} = {1100101}
{110} * {10111} = {1110010}
{001} * {10111} = {0010111} The products are closed under modulo-two addition.

The second order polynomial representations relate to Octonion multiplication characteristics, demonstrated by closed set modulo-two addition operations. Since this closed set addition operation carries over to the cyclic product codes, it is reasonable to look for some connection between Octonion multiplication and these cyclic codes. The following are the products of the second order polynomials with each of the generator polynomials arranged in the singular cyclic and antisymmetric order they produce.

Generator 1
{100} * {11101} = {1 1 1 0 1 0 0}
{010} * {11101} = {0 1 1 1 0 1 0}
{001} * {11101} = {0 0 1 1 1 0 1}
{110} * {11101} = {1 0 0 1 1 1 0}
{011} * {11101} = {0 1 0 0 1 1 1}
{111} * {11101} = {1 0 1 0 0 1 1}
{101} * {11101} = {1 1 0 1 0 0 1}

Generator 2
{101} * {10111} = {1 0 0 1 0 1 1}
{111} * {10111} = {1 1 0 0 1 0 1}
{110} * {10111} = {1 1 1 0 0 1 0}
{011} * {10111} = {0 1 1 1 0 0 1}
{100} * {10111} = {1 0 1 1 1 0 0}
{010} * {10111} = {0 1 0 1 1 1 0}
{001} * {10111} = {0 0 1 0 1 1 1}

The arrangements must be antisymmetric to represent the unit anti-commutation multiplication properties of unlike vectors. The same row unit progression top to bottom is transferred to the columns left to right. This gives the following as a representation of the Octonion vector unit multiplication table from each of the generator polynomials.

	4	2	1	6	3	7	5				
4	{1	1	1	0	1	0	0}	Generated Algebra			
2	{0	1	1	1	0	1	0}				
1	{0	0	1	1	1	0	1}		(123)	(761)	(145)
6	{1	0	0	1	1	1	0}			(572)	(246)
3	{0	1	0	0	1	1	1}			(653)	(347)
7	{1	0	1	0	0	1	1}				
5	{1	1	0	1	0	0	1}	Left Octonion Algebra			

	5	7	6	3	4	2	1				
5	{1	0	0	1	0	1	1}	Generated Algebra			
7	{1	1	0	0	1	0	1}				
6	{1	1	1	0	0	1	0}		(563)	(725)	(541)
3	{0	1	1	1	0	0	1}			(176)	(642)
4	{1	0	1	1	1	0	0}			(213)	(347)
2	{0	1	0	1	1	1	0}				
1	{0	0	1	0	1	1	1}	Left Octonion Algebra			

The modulo-two addition determines the unit number of the product. Looking at the 1 vs. 0 for like unit products, the table value is always 1. The polarity of this product is always –1, so perhaps we may associate a product polarity of –1 for every 1, and +1 for every 0. Doing this, we arrive at the permutations indicated above, valid for what I call Left Octonion Algebra. The association with Left and Right for a representation of **O** algebra will be clarified below.

Now there was nothing forcing the association of the code word – generator product with rows as was done above. It could have just as well been associated with columns of the multiplication table. This would produce the transpose of the multiplication tables above. These become multiplication tables for Right Octonion Algebra.

There are no generators from roots of the polynomial $p^{15}+1$ that produce suitable cyclic codes that can be arranged in the required 1/0 antisymmetric fashion to yield proper unlike unit product anti-commutation. Perhaps there is a connection between this fact and the fact that the quality of being a normed hypercomplex division algebra does not extend to the 16 dimensional sedenions.

Addressing now the Quaternions, units 1, 2 and 3 are mapped to the polynomials {01}, {10} and {11} respectively. The roots of the polynomial p^3+1 are:

{1001} = {11}*{111}

The single generator for cyclic codes is {11}. The antisymmetric arrangement is

```
                  2  1  3
2: {10}*{11} = {1  1  0}
1: {01}*{11} = {0  1  1}
3: {11}*{11} = {1  0  1}
```

This generates the permutation (123), and the vector unit multiplication table for H.

Creation of all Octonion Algebras

Next, an examination of all possible valid Octonion permutation sets. I will exclusively use the 7 triplet groupings from above. Remember, the names associated with positions within the resultant permutations are arbitrary names only, and not definitions of the algebra. Any new and different grouping of names, like using (124) instead of (123) is nothing more than an alias.

It is assumed that the scalar-vector and like vector product rules mentioned above are requirements of all possible **O** algebras. Then the only algebraic modification possible is permutation negation, which is simply achieved by swapping the positions of two permutation members. Since any second swap returns the permutation to its original definition, there are only two possibilities for each permutation.

Two possibilities on seven permutations yield 128 possible candidates for Octonion algebras. Each vector unit is a member of 3 different permutation groups, so unlike the Quaternions, we cannot assume the negation of any number of permutations is an isomorphism, or even that it results in a valid Octonion algebra. Each of the 128 possibilities must be vetted by some rule. A suitable rule is Artin's Rule, which states a limited 3-product associativity under multiplication for the generally non-associative Octonion algebra. This defines the following test for our candidates:

For all 8 dimensional hypercomplex a and b

/a * (a*b) ?= (/a * a) * b

Here /a is the conjugate of a, and "*" is defined by the **O** candidate algebra multiplication rules.

Only 16 of the 128 candidates answer this test in the affirmative, and are therefore valid Octonion algebras. Anticipating their Left/Right characteristics, the following columns represent the 16 valid permutation sets for the Octonion Algebras.

Left **O** Algebra

Column	0	1	2	3	4	5	6	7
	(123)	(123)	(123)	(123)	(321)	(321)	(321)	(321)
	(761)	(761)	(167)	(167)	(167)	(167)	(761)	(761)
	(572)	(275)	(572)	(275)	(275)	(572)	(275)	(572)
	(653)	(356)	(356)	(653)	(356)	(653)	(653)	(356)
	(145)	(145)	(541)	(541)	(145)	(145)	(541)	(541)
	(246)	(642)	(246)	(642)	(246)	(642)	(246)	(642)
	(347)	(743)	(743)	(347)	(347)	(743)	(743)	(347)

Right **O** Algebra

Column	0	1	2	3	4	5	6	7
	(123)	(123)	(123)	(123)	(321)	(321)	(321)	(321)
	(761)	(761)	(167)	(167)	(167)	(167)	(761)	(761)
	(572)	(275)	(572)	(275)	(275)	(572)	(275)	(572)
	(653)	(356)	(356)	(653)	(356)	(653)	(653)	(356)
	(541)	(541)	(145)	(145)	(541)	(541)	(145)	(145)
	(642)	(246)	(642)	(246)	(642)	(246)	(642)	(246)
	(743)	(347)	(347)	(743)	(743)	(347)	(347)	(743)

The left/right handedness can be seen within the groups in the following way. Pick any column of any group. Within the chosen column pick any single unit and cyclically rotate the 3 permutations the chosen unit belongs to such that the selected unit is in the central position. For any Right **O** Algebra, the 3 units on the right end of the triplets will be members of another permutation and the 3 on the left will not. Likewise for any Left **O** Algebra, the 3 units on the left end of the triplets will be the members of another permutation and the ones on the right will not. This is an intrinsic characteristic of Octonion Algebra. Clearly, Right and Left Octonion Algebra have no possible mapping between the two defined by a consistent renaming of units, since the intrinsic chiral property would map in kind. Therefore, Right Octonion Algebra is not isomorphic to Left Octonion Algebra.

The isomorphic mapping between like handed Octonion Algebras is indicated within each of the columns above. Taking column 0 as the prototype, column 1 negates the 4 permutations of the prototype that do not contain unit 1. Likewise, column n has the 4 permutations of the prototype not containing unit n negated. Any column maps to any of the other seven of the same handedness by negating the 4 permutations that do not include 1 of the 7 units.

One possible non-isomorphic mapping between left and right is also indicated above. Column n for each algebra type differs from the same column of the other type by a negation of all permutations containing the unit 4. The map could have equally been negating the 3 permutations that include any other unit. The chiral change map is thus negating 3 permutations that share a common unit. Each column of either type maps to 1 of the 8 in the other type by either negating all permutations, or negating the 3 permutations that include 1 of the 7 units.

Any combination of chiral change/no change mappings may be applied to any of the 16 algebras defined above. No matter what the combination, the result will be one of the above. Any single mapping operation that does not negate either the 3 permutations that include one of the units, or the 4 permutations that do not include one of the units, will create an algebra that is not valid Octonion.

I take as the prototypes the left most columns above, restated here

Left **O** Algebra

Set Ia	Set Ib	Set II
(123)	(761)	(145)
	(572)	(246)
	(653)	(347)

Right **O** Algebra

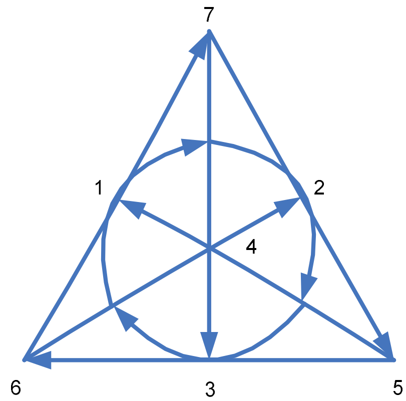
Set Ia	Set Ib	Set II
(123)	(761)	(541)
	(572)	(642)
	(653)	(743)

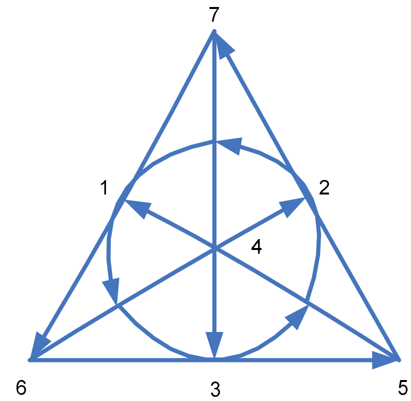
Pictorial Representations of Octonion Algebras

There are two published representations that I am familiar with, neither of which have covered both types of Octonion Algebras when in fact they could. One is the Fano Projective Plane representation is in the Wikipedia explanation of Octonion multiplication [5] and has also been mentioned by John Baez in “The Octonions” [6]. The other was a three-handed pinwheel representation in “The Road to Reality” by Roger Penrose [7]. Each source showed only a single algebra representation.

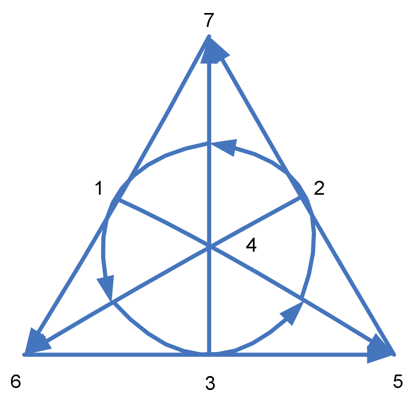
The Fano Plane representation actually has four different valid forms, two for each hand of the algebra. It all comes down to the arrow directions. There are seven arrowed paths indicating a flow through three numbered points. These represent the seven permutations of the algebra, indicating triplet and order within the permutation. All valid representations will have all three arrows on the triangle perimeter and the enclosed circle arrow going all clockwise or all counter-clockwise. The switch between them is the isomorphic map for **O** algebra, it does not change the chiral sense. The three triangle vertex angle bisector arrows will all point away or all point towards the vertex. The switch between is the non-isomorphic map, it changes between left and right sense.

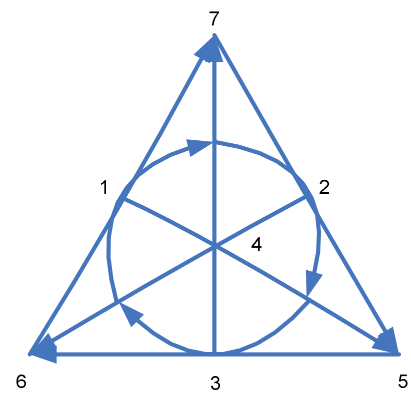
Right Octonion Algebra is represented in the Fano Plane with each vertex angle bisector pointing away from the vertex. The two isomorphic views are these.



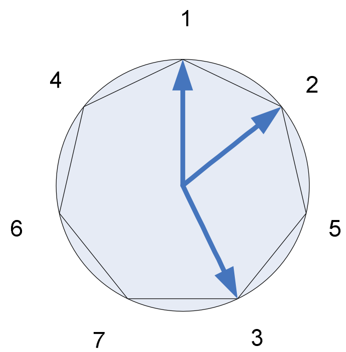


Left Octonion Algebra is represented by a Fano Plane view with each bisector arrow pointing towards the vertex. The two isomorphic views are these.

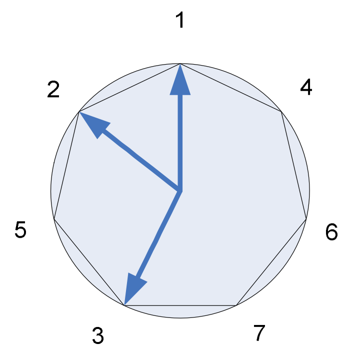




The pinwheel approach has the seven Octonion unit numbers equally spaced around the perimeter of a circle. Inside the circle are three clock hands with fixed spacing of 1-2-4 perimeter units. The unit numbers pointed to by the hands are the three choices for a permutation, and the order within is read off going clockwise. The three clock hands move in fixed juxtaposition picking off all seven permutations. One way of achieving the non-isomorphic map between algebras is to simply change the direction around the perimeter when reading off the three permutation units. Another way would be to stay with clockwise direction order of triplets but change the 1-2-4 spacing direction and perimeter number direction. This is done below to indicate Left Octonion Algebra and Right Octonion Algebra.



Right Octonion Algebra



Left Octonion Algebra

Octonion Product Variance When Changing Algebra

Now investigate the results of algebraic **O** expressions in terms of how they vary with the selection of a particular algebra amongst the 16 possibilities. Any algebraic expression with at least one **O** product will result in a sum of positive or negative product terms in the coefficients of the constituents. Since the sixteen algebras do not modify the triple of units in any permutation, only their order within, the set of resultant product terms will not change coefficient pairings if the algebra is changed. The only thing that might change is the sign of select product terms. Starting with initial constituents that are all individually invariant, and a particular choice of algebra, there are algebraic expressions in **O** for which the signs of all product terms in the result remain the same when the algebra is changed to any one of the other fifteen related algebras. These expressions are algebraic invariants. There will also be expressions in **O** that have a combination of invariant and non-invariant product terms, or all non-invariant. These expressions are algebraic variants.

While it is strictly speaking not proper to combine a results from one algebra with results from another, the operations of **O** addition and subtraction may be used in the case of related **O** algebras to separate result product terms that change sign from those that do not when the algebra is changed. For a given expression let "Ra" be the result using **O** algebra "a" rules and "Rb" be the result using **O** algebra "b" rules. The expression (Ra – Rb)/2 will be the sum of Ra product terms that change sign when the algebra changes from a to b. Likewise the expression (Ra + Rb)/2 will be the sum of Ra product terms that do not change sign when the algebra changes from a to b. For a given expression in which all of the initial constituents are algebraic invariants, define for the expression the "distance" between its representations in algebras "a" and "b" to be (Ra – Rb) /2. An expression is an explicit algebraic invariant if the distance function for any choice of paired algebras is identically zero. If an expression is expected by principle to be an algebraic invariant, invariance may be forced by equating to zero every distance function not explicitly zero. This forcing may be construed as assigning equations of constraint.

A distance function created by comparing expression results using the rules of two **O** algebras is not necessarily independent of the distance function created with a different pair of **O** algebras. There will be product terms in each of these distance functions that will also appear in other distance functions.

Define the distance between representations in any two of sixteen possible **O** algebras as a "long distance". The full set of distance functions between **O** algebras within a single Left or Right type can be reduced to a set of seven distances I define as "short distances". The seven short distances for a given Left or Right **O** Algebra type are unique and independent, there are no repeated product terms between them. The short distances for Right **O** Algebra may or may not be identical to the short distances for Left **O** Algebra, it depends on the expression in question. More will be said about this below. The short distances can be derived for either Left or Right **O** Algebras with the following sieve process.

The Octonion Variance Sieve Process

As indicated above, the eight members of a given algebra type are formed by picking some valid Octonion permutation representation as a starting point, the prototype, then forming seven more algebra representations by negating the four permutations in the prototype that do not include each of the seven vector unit choices in turn. Call these algebras O_n, where n=0 for the original prototype Left **O** Algebra choice, and for 0<n<8, let O_n represent the algebra formed by negating the four permutations of O_0 that do not include the unit n.

Create any arbitrary algebraic expression constructed from invariant constituents. Let the result using the rules of algebra O_n be $R[O_n]$. The algebraic invariant component of result R and each of its seven short distances will be a linear combination of the eight representations $R[O_n]$. Each of the short distances will be related to one of the permutation groups in the following way. Let SL(pqr) be the short distance associated with the Left **O** permutation (pqr), and let the Left algebraic invariant portion of the result be IL. Then

$SL(123) = 1/8 \{R[O_0] + R[O_1] + R[O_2] + R[O_3] - R[O_4] - R[O_5] - R[O_6] - R[O_7] \}$
$SL(761) = 1/8 \{R[O_0] + R[O_7] + R[O_6] + R[O_1] - R[O_2] - R[O_3] - R[O_4] - R[O_5] \}$
$SL(572) = 1/8 \{R[O_0] + R[O_5] + R[O_7] + R[O_2] - R[O_1] - R[O_3] - R[O_4] - R[O_6] \}$
$SL(653) = 1/8 \{R[O_0] + R[O_6] + R[O_5] + R[O_3] - R[O_1] - R[O_2] - R[O_4] - R[O_7] \}$
$SL(451) = 1/8 \{R[O_0] + R[O_4] + R[O_5] + R[O_1] - R[O_2] - R[O_3] - R[O_6] - R[O_7] \}$
$SL(462) = 1/8 \{R[O_0] + R[O_4] + R[O_6] + R[O_2] - R[O_1] - R[O_3] - R[O_5] - R[O_7] \}$
$SL(473) = 1/8 \{R[O_0] + R[O_4] + R[O_7] + R[O_3] - R[O_1] - R[O_2] - R[O_5] - R[O_6] \}$
$IL = 1/8 \{R[O_0] + R[O_1] + R[O_2] + R[O_3] + R[O_4] + R[O_5] + R[O_6] + R[O_7] \}$

So a short distance is found by adding the result in the prototype algebra to the results using the algebras formed by isomorphic mapping the prototype algebra around each of the three units of a selected permutation, then subtracting each of the remaining results for the algebras formed by isomorphic mapping around the units not found in that permutation, then dividing this result by eight. The full algebraic expression $R[O_0]$ is seen to be the sum of the algebraic invariant portion and each of these seven short distances.

For each of the Left **O** algebras, there is a companion Right **O** algebra that can be created by negating all seven permutations in the selected Left **O** representation. Define the companion pairings to be $O_n \rightarrow O_{n+8}$, where n runs 0 to 7. Define companion short distances and algebraic invariant for Right **O** algebra by adding 8 to each of the Left **O** indices for SL(pqr) and IL:

$SR(321) = 1/8 \{R[O_8] + R[O_9] + R[O_{10}] + R[O_{11}] - R[O_{12}] - R[O_{13}] - R[O_{14}] - R[O_{15}] \}$
$SR(167) = 1/8 \{R[O_8] + R[O_{15}] + R[O_{14}] + R[O_9] - R[O_{10}] - R[O_{11}] - R[O_{12}] - R[O_{13}] \}$
$SR(275) = 1/8 \{R[O_8] + R[O_{13}] + R[O_{15}] + R[O_{10}] - R[O_9] - R[O_{11}] - R[O_{12}] - R[O_{14}] \}$
$SR(356) = 1/8 \{R[O_8] + R[O_{14}] + R[O_{13}] + R[O_{11}] - R[O_9] - R[O_{10}] - R[O_{12}] - R[O_{15}] \}$
$SR(154) = 1/8 \{R[O_8] + R[O_{12}] + R[O_{13}] + R[O_9] - R[O_{10}] - R[O_{11}] - R[O_{14}] - R[O_{15}] \}$
$SR(264) = 1/8 \{R[O_8] + R[O_{12}] + R[O_{14}] + R[O_{10}] - R[O_9] - R[O_{11}] - R[O_{13}] - R[O_{15}] \}$
$SR(374) = 1/8 \{R[O_8] + R[O_{12}] + R[O_{15}] + R[O_{11}] - R[O_9] - R[O_{10}] - R[O_{13}] - R[O_{14}] \}$
$IR = 1/8 \{R[O_8] + R[O_9] + R[O_{10}] + R[O_{11}] + R[O_{12}] + R[O_{13}] + R[O_{14}] + R[O_{15}] \}$

Except possibly for signs, the product term sets for SL(pqr) and SR(rqp) will be the same. If SL(pqr)=SR(rqp), then their equivalent short distances are irreducible. Define an irreducible distance as a “minimum distance”. If the signs of the product terms in SL(pqr) and SR(pqr) are not all the same, then the two companion short distances reduce to two separate minimum distances given by ½[SL(pqr)+SR(rqp)] and ½[SL(pqr)-SR(rqp)].

If an algebraic expression is forced to be an algebraic invariant by equating all possible distance functions to zero, this becomes equivalent to equating each of the minimum distance functions to zero. The relationship IL=IR can be demonstrated to hold for any expression.

If Octonion Algebra is to be successful in describing physical phenomena, then the **O** representation of physical reality must be algebraically invariant. This is a logical conclusion once we accept the belief that there is no preferred algebra amongst the 16 possibilities, since using one preferred choice would be the only way to get a consistent result. This belief should be easily accepted after the detailed and complete analysis of the algebras done above, where there is absolutely nothing to distinguish one algebra as being more likely than another. I therefore state the following as Law.

The Law of Octonion Algebraic Invariance

The mathematical form of any physical phenomenon accurately described by Octonion algebra is algebraically invariant.

Now for the connection between Octonion Algebra and the physics of reality.

Connecting Electrodynamics To The Algebra of Octonions

In Electrodynamics, the electric and magnetic field components are separately maintained in the second rank field tensor. This happens for a good reason, the two fields are fundamentally different, preventing them from occupying the same vector space. It is normal to assign the qualitative names of axial and polar to these field types. If we examine the polarity and physical orientation of the fields when the 3D physical space is changed from a right handed system to a left handed system, we find the axial field will not change sign, but the polar field will change sign. Therefore the magnetic field will point in the opposite physical direction when the handedness of the 3D system is changed, it is an algebraic variant. The electric field will not change its physical direction when the handedness of the 3D system is changed; it is an algebraic invariant.

The orientation characteristics for axial and polar vector types carry through to their possible vector product combinations. It can be shown that the vector product of two polar vectors matches the characteristics of an axial vector when the system handedness changes, the vector product of two axial vectors retains axial vector characteristics. The vector product of an axial and polar vector matches polar vector characteristics. If an algebra is to be associated with these field types, it must faithfully represent their product characteristics.

In both Left and Right **O** algebra types as shown above, we can associate the familiar physical 3D rectangular (xyz) axes with the unit triplets 123 and 567. Axial vector types are closed under the operation of vector multiplication. Any single permutation group like (123) is likewise closed under multiplication. Units 1,2 and 3 of the single Set Ia permutation of the chosen Right and Left **O** Algebra prototypes above may therefore be associated with the axial vector units corresponding to x, y and z respectively. The vector product of two polar vectors matches axial vectors, and the vector product of a polar vector and an axial vector matches polar vectors. The three permutations of Set Ib in the chosen Left and Right Octonion Algebra prototypes above properly indicate this multiplication scheme with units 5,6 and 7 representing the polar vector units corresponding to x, y and z respectively.

One would then expect to find the magnetic field components as rotational constructs in Octonion units 1, 2 and 3. Likewise, the electric field components should be present as irrotational constructs in Octonion units 5, 6 and 7. It is possible to create axial (rotational) and polar (irrotational) constructs with any of the 7 Octonion vector units, and algebraic variant and invariant forms within all Octonion units. We must therefore not take Octonion units 1, 2 and 3 to be a basis exclusively for non-invariant types, or 5,6 and 7 exclusively for invariant. The salient feature is not axial vs. polar, or physical orientation properties under 3D handedness change, it is the multiplication rules, and therefore the structure of the applied algebra. The Octonion Algebras faithfully produce the expected multiplication rules for the fields of Electrodynamics by design.

Electrodynamics is one of the best understood areas of study in physics. The path to understanding Electrodynamics led to the breakthrough understanding that the proper mathematical formalism for physical reality must combine scalar and vector holistically. This led to the now familiar concepts of 4D space-time and relativity. Electrodynamics will again be used here as a roadmap to show the way within the framework of Octonion Algebra. We will look for familiar forms from Electrodynamics, and see what else may be coming along for the ride. But first, there must be a foundation laid describing a proper formalism for Octonion differential calculus. The presentation applies equally to the algebra of Quaternions, but our later focus will be on Octonion Algebra, since it is only within **O** that Electrodynamics is adequately represented. Summation convention as assumed on repeated indices unless otherwise stated.

The Calculus of H and O Algebras

Define a space of either four or eight dimensions. Attach to these an algebraic structure such that the operation of vector multiplication can be defined consistent with **H** or **O** algebra. Define the rectilinear position vector $r = r_j u_j$ a member of **H** or **O**. The coordinates r_j must all have the same dimensional classification (distance: meters, feet, etc.) so that all of the product terms resulting from the operation of multiplication of two vectors will likewise be consistent in type.

Define a general curvilinear basis vector set for **H** or **O** algebra by a diffeomorphism on the rectilinear basis by implying a functional relationship between the rectilinear r and a new set of variables v_k where k runs 0 to 3 for **H** algebra and 0 to 7 for **O** algebra. The intrinsic **H** or **O** units u_j are taken to be constants independent of any possible set v. Then the **H** or **O** curvilinear basis vector set will be defined as

$w_i = \partial/\partial v_i\ [r_j(v)]\ u_j.$

The unit types for the variable set v need not all be the same type, since the **H** or **O** algebra and its units are only indirectly seen by the v system through the functional relationship between v and r. An example would be spherical-polar **H** algebra where v would encompass time, distance, and two angles. The functional relationships will resolve any required normalization needed due to variation in v variable types.

The Jacobian matrix elements J_{ij} are the coefficients $\partial r_j/\partial v_i$ of the diffeomorphism and the Jacobian J = determinant [J_{ij}] is taken to not equal 0 as usual.

Define the differential of the position vector to be

$dr = w_i\ dv_i$, a member of **H** or **O**.

Then the **H** or **O** product

$dr * /dr = ds^2 = g_{ab}\ dv_a\ dv_b$,

where (/) denotes conjugation and $g_{ab} = \langle w_a,w_b\rangle = g_{ba}$ is the metric matrix. The inner product may be cast in **H** or **O** algebra as $½\ (/w_a * w_b + /w_b * w_a)$.

Form a matrix from the intrinsic **H** or **O** coefficients of dr vectors

$dr_{ij} = (\partial r_j /\partial v_i)\ dv_i$.

Define the differential hypervolume element as

dV = determinant [dr_{ij}.].

Define the differential hypersurface normal vector set as the **H** or **O** row vectors dS_i of the matrix

dS_{ij} = cofactor(ij) of dr_{ij}.

That is

$dS_i = dS_{ij}\ u_j$.

For any general curvilinear system, the full complement of traditional vector differentiation forms (divergence, gradient and curl) as well as properly formulated scalar differentiation products for left differentiation operations (non-commutative algebras here) on the **H** or **O** vector P are included in the left ensemble derivative defined as

$E(P) = \text{limit as} \int dV \to 0 \quad \sum_j : \int [dS_j * P] / \int dV$

where * is **H** or **O** multiplication, and the surface integral in the numerator is over the enclosed volume prescribed in the denominator.

This integral definition is simply an extension of the familiar classical 3D integral definitions for divergence, gradient and curl. The **H** algebra product embodies all of these in a holistic way, perhaps more fundamentally. The extension to **O** algebra is natural, with multiple divergence, gradient and curl forms indicative of the multiple **H** sub-algebras, alternatively viewed collectively as an **O** form in its own right.

The full complement of right differentiation operations on **H** or **O** vector P is included in the right ensemble derivative defined as

$(P)E = \text{limit as} \int dV \to 0 \quad \sum_j : \int [P * dS_j] / \int dV$

The left and right ensemble forms are equivalent to

$E(P) = 1/J \;\; \partial/\partial v_i \;\; (N_i * P)$ and

$(P)E = 1/J \;\; \partial/\partial v_i \;\; (P * N_i)$ where

$N_i = dS_i \; dv_i \; / (dv_0 \, dv_1 \, dv_2 \ldots.. dv_n)$,

with n=3 for H algebra and n=7 for **O** algebra, and J is the Jacobian of the diffeomorphism. The partial differentiation is a scalar operation on its argument. The **H** or **O** vector P may be expressed as $P_j \; w_j$, using the covariant components for the basis w, or equivalently with the intrinsic algebra basis as $A_k \; u_k$.

The diffeomorphism is by definition invertible, with representations in terms of $\partial v_i/\partial r_j$ instead of $\partial r_j/\partial v_i$, where the Jacobian J from above is associated in the expression

$J = 1 / \text{determinant} [\partial v_i/\partial r_j]$ and the orthogonality conditions are

$\partial v_i/\partial r_j \;\; \partial r_j/\partial v_k = \partial v_r/\partial r_i \;\; \partial r_k/\partial v_r = \delta_{ik}$

These two come about from the requirements that r be a function of v, and conversely v be a function of r, and that the components of either r or v be functionally independent, stated as

$\partial r_i/\partial r_k = \delta_{ik}$ and $\partial v_i/\partial v_k = \delta_{ik}$

If the inverted forms $\partial v_i/\partial r_j$ are readily available, the surface normal expression may be simply written as

$N_i = J\ \partial v_i/\partial r_j\ u_j$.

This allows the ensemble differentiation form to be written as

$E(P) = 1/J\ \partial/\partial v_i\ (J\ \partial v_i/\partial r_j\ P_n\ \partial r_m/\partial v_n)\ (u_j * u_m)$

and

$(P)E = 1/J\ \partial/\partial v_i\ (J\ \partial v_i/\partial r_j\ P_n\ \partial r_m/\partial v_n)\ (u_m * u_j)$

Otherwise, the general e system forms of the ensemble derivatives come about as follows.

For **O** algebra

$N_A = 1/7!\ e_{IJKLMNPQ}\ e_{ABCDEFGH}\ \partial r_J/\partial v_B\ \partial r_K/\partial v_C\ \partial r_L/\partial v_D\ \partial r_M/\partial v_E\ \partial r_N/\partial v_F\ \partial r_P/\partial v_G\ \partial r_Q/\partial v_H\ u_I$

For **H** algebra

$N_A = 1/3!\ e_{IJKL}\ e_{ABCD}\ \partial r_J/\partial v_B\ \partial r_K/\partial v_C\ \partial r_L/\partial v_D\ u_I$

$P = P_S\ \partial r_T/\partial v_S\ u_T$ where S,T run 0-3 for H and 0-7 for O

E(P) for **O** is then

$1/(7!\ J)\ e_{IJKLMNPQ}\ e_{ABCDEFGH}$
$\partial/\partial v_A\ [\ \partial r_J/\partial v_B\ \partial r_K/\partial v_C\ \partial r_L/\partial v_D\ \partial r_M/\partial v_E\ \partial r_N/\partial v_F\ \partial r_P/\partial v_G\ \partial r_Q/\partial v_H\ \partial r_T/\partial v_S\ P_S\ ]\ \ [\ u_I\ u_T\ ]$

E(P) for **H** is then

$1/(3!\ J)\ e_{IJKL}\ e_{ABCD}\ \partial/\partial v_A\ [\ \partial r_J/\partial v_B\ \partial r_K/\partial v_C\ \partial r_L/\partial v_D\ \partial r_T/\partial v_S\ P_S\ ]\ \ [\ u_I\ u_T\ ]$

The forms for (P)E are these but with the multiplication of the intrinsic algebraic units commuted.

In the trivial rectilinear case of r = v with $r_0 = ct$, $v_0 = t$ (time) and c is the speed of light

$\partial r_i/\partial v_k = \delta_{ik}$ and $\partial v_i/\partial r_k = \delta_{ik}$ for ik not 0, and

$P_i = A_i$ also

$J = c$

This allows the ensemble forms to be written as

$E(P) = \partial A_j/\partial r_i \quad u_i \ * u_j$ and

$(P)E = \partial A_j/\partial r_i \quad u_j \ * u_i$

Here $\partial/\partial r_0 = 1/c\ \partial/\partial t$.

These two can be expressed in simple differential operator notation as

$E(P) = u_i\ \partial/\partial r_i > [A_j\ u_j] = D^*(A)$

$(P)E = [A_j\ u_j] < /u_i\ \partial/\partial r_i \} = (A)^*D$

Here the < and > are indicating the direction of application of the partial differentiation and algebraic multiplication of the included units. In this operator form, D is treated like any other **H** or **O** vector, that is D as $u_n\ D_n$. Its application is through the operation of multiplication as defined by the algebra followed by the scalar differentiation operation.

The Octonion Fields

When the ensemble derivative is applied to the 8-potential vector A_k, the result will be all possible fields described in the framework of Octonions. The structure of this ensemble form is best visualized in rectilinear coordinates, without all of the necessary clutter of a curvilinear system. It will of course take on a mix of algebraic invariants in each irrotational form, and algebraic variants in each rotational form. It could alternatively be formed by a right application of the ensemble derivative, or a left application. We could use Left **O** Algebra or Right **O** algebra. It would not be prudent to insist on algebraic invariance in the field expression, since the fields themselves are not physically observable. The observables are the manifested forces. For result unit [n] using prototype Right **O** algebra, the D*(A) result is

F =
[0] D0(A0)-D1(A1)-D2(A2)-D3(A3)-D4(A4)-D5(A5)-D6(A6)-D7(A7)
[1] [D0 (A1)+D1 (A0)] + [D2 (A3)-D3 (A2)] + [D5 (A4)-D4 (A5)] + [D7 (A6)-D6 (A7)]
[2] [D0 (A2)+D2 (A0)] + [D3 (A1)-D1 (A3)] + [D6 (A4)-D4 (A6)] + [D5 (A7)-D7 (A5)]
[3] [D0 (A3)+D3 (A0)] + [D1 (A2)-D2 (A1)] + [D7 (A4)-D4 (A7)] + [D6 (A5)-D5 (A6)]
[4] [D0 (A4)+D4 (A0)] + [D1 (A5)-D5 (A1)] + [D2 (A6)-D6 (A2)] + [D3 (A7)-D7 (A3)]
[5] [D0 (A5)+D5 (A0)] + [D7 (A2)-D2 (A7)] + [D4 (A1)-D1 (A4)] + [D3 (A6)-D6 (A3)]
[6] [D0 (A6)+D6 (A0)] + [D5 (A3)-D3 (A5)] + [D4 (A2)-D2 (A4)] + [D1 (A7)-D7 (A1)]
[7] [D0 (A7)+D7 (A0)] + [D6 (A1)-D1 (A6)] + [D4 (A3)-D3 (A4)] + [D2 (A5)-D5 (A2)]

I take the last three bracketed terms in units [1], [2] and [3] to be the (-)magnetic field components for the physical 3D components x, y and z respectively. The (-)electric field components will be the first bracketed terms in units [5], [6] and [7]. I will presume the first bracketed terms in units [1], [2] and [3] are the x, y and z physical components of the (-)gravitational field. The other bracketed terms are additional fields for which I have no names. The scalar term in [0] will be algebraically removed from consideration such that we do not have a drag or acceleration force on the current density when we later form the Action Function, which will be produced from the product of the 8-current and the field expression.

The treatment of time here needs some emphasis in regards to the difference between this presentation and the ideas of special and general relativity. In relativity, the domain for time is considered to be solely the scalar dimension of the 4D position vector. This is the case for both stationary and moving frames of reference.

In the ensemble formalism, time is taken to be an independent variable always of the v side of the diffeomorphism off of the **O** intrinsic r representation. The notions of scalar and vector are only directly and singularly attached to the intrinsic r coordinates of the algebra. A one-to-one association between the intrinsic scalar and vector units to each of the curvilinear v coordinates is not implied. Just as with the diffeomorphism off rectangular coordinates to spherical-polar coordinates, the v system basis vectors may each be composed of multiple u_n components. If v_0 is assumed to be time, then $\partial r_i/\partial v_0\ u_i$, the basis vector for v_0 is pure scalar only if the vector components of r are stationary in time, i.e. no apparent velocity between the r and v systems. In this case, time is indeed pure scalar. As soon as vector r becomes a function of time, i.e. a velocity transformation to v, the basis vector for v_0 picks up non-scalar components weighted by the velocity of the transformation. Time is no longer just a scalar form. In the ensemble formalism, the vector components of the time basis vector manifest the added terms of what is classically called the convective derivative.

The Octonion 8-Current

Higher order differential equations are created by additional applications of the ensemble form. Following the Electrodynamics roadmap, we should next look for a form representing the 8-current. With this in hand, the next step is to form the vector forces and scalar work representing the Action Function. When the Action Function is directly integrated over the 7-volume resulting in a surface integral representation, the result will be the analogy of the integration of the divergence of the Electrodynamics stress-energy-momentum tensor representative of the conservation laws for energy and momentum.

When the process outlined below integrates the Action Function, the result will indicate that an additional form must be added to the expected **O** extension to the familiar D’Alembertian formation of the 4-current. In the derivation of the 4-current of Electrodynamics, there is a similar form that is dispatched through the Lorentz Gauge, called the Lorentz Condition. The following algebraic expression for the 8-current includes both the anticipated 8-dimensional extension of the D’Alembertian scalar operation on the 8-potential, and an additional 8-gradient of a scalar form we could call the Analogous Lorentz Condition.

Then j = 1/4 (E[(/A)/E - E(A)] + [/E(/A) - (A)E]E)

This is left E on the vector only left field term added to right E on the vector only right field.

Written out in rectilinear coordinates, j =

[0]
-D0^2(A0)+D1^2(A0)+D2^2(A0)+D3^2(A0)+D4^2(A0)+D5^2(A0)+D6^2(A0)+D7^2(A0)
+D0 [D0 (A0)+D1 (A1)+D2 (A2)+D3 (A3)+D4 (A4)+D5 (A5)+D6 (A6)+D7 (A7)]
[1]
-D0^2(A1)+D1^2(A1)+D2^2(A1)+D3^2(A1)+D4^2(A1)+D5^2(A1)+D6^2(A1)+D7^2(A1)
-D1 [D0 (A0)+D1 (A1)+D2 (A2)+D3 (A3)+D4 (A4)+D5 (A5)+D6 (A6)+D7 (A7)]
[2]
-D0^2(A2)+D1^2(A2)+D2^2(A2)+D3^2(A2)+D4^2(A2)+D5^2(A2)+D6^2(A2)+D7^2(A2)
-D2 [D0 (A0)+D1 (A1)+D2 (A2)+D3 (A3)+D4 (A4)+D5 (A5)+D6 (A6)+D7 (A7)]
[3]
-D0^2(A3)+D1^2(A3)+D2^2(A3)+D3^2(A3)+D4^2(A3)+D5^2(A3)+D6^2(A3)+D7^2(A3)
-D3 [D0 (A0)+D1 (A1)+D2 (A2)+D3 (A3)+D4 (A4)+D5 (A5)+D6 (A6)+D7 (A7)]
[4]
-D0^2(A4)+D1^2(A4)+D2^2(A4)+D3^2(A4)+D4^2(A4)+D5^2(A4)+D6^2(A4)+D7^2(A4)
-D4 [D0 (A0)+D1 (A1)+D2 (A2)+D3 (A3)+D4 (A4)+D5 (A5)+D6 (A6)+D7 (A7)]
[5]
-D0^2(A5)+D1^2(A5)+D2^2(A5)+D3^2(A5)+D4^2(A5)+D5^2(A5)+D6^2(A5)+D7^2(A5)
-D5 [D0 (A0)+D1 (A1)+D2 (A2)+D3 (A3)+D4 (A4)+D5 (A5)+D6 (A6)+D7 (A7)]
[6]
-D0^2(A6)+D1^2(A6)+D2^2(A6)+D3^2(A6)+D4^2(A6)+D5^2(A6)+D6^2(A6)+D7^2(A6)
-D6 [D0 (A0)+D1 (A1)+D2 (A2)+D3 (A3)+D4 (A4)+D5 (A5)+D6 (A6)+D7 (A7)]
[7]
-D0^2(A7)+D1^2(A7)+D2^2(A7)+D3^2(A7)+D4^2(A7)+D5^2(A7)+D6^2(A7)+D7^2(A7)
-D7 [D0 (A0)+D1 (A1)+D2 (A2)+D3 (A3)+D4 (A4)+D5 (A5)+D6 (A6)+D7 (A7)]

The 8-current j is an algebraic invariant as expected, since it is a physical observable. The Analogous Lorentz condition is

[D0 (A0)+D1 (A1)+D2 (A2)+D3 (A3)+D4 (A4)+D5 (A5)+D6 (A6)+D7 (A7)] = 0

The Octonion Action Function

The next milestone on the Electrodynamics roadmap is to come up with an expression for the Action Function. This must indicate the concept of work in the scalar term, and all forces in the vector terms. An obvious first guess would be the Octonion product of the 8-current and the field form, since it will surely produce the scalar product of current density and electric field, the product of charge density and electric field, as well as the vector product of current density and the magnetic field. The 8-current is an algebraic invariant and the field expressions from left or right differentiation of the potentials in the field form F are mixed invariant and variant. The simple left or right product of j with either the left or right field definition is assured to contain both algebraic invariants and variants. The work and force functions must be fully invariant since they are physical observables.

It will be informative to run F * j through the variance sieve outlined above. The results are as follows.

Invariant form = ½ (IL + IR) == The Action Function

[0]
-j1 [D0(A1)+D1(A0)]
-j2 [D0(A2)+D2(A0)]
-j3 [D0(A3)+D3(A0)]
-j4 [D0(A4)+D4(A0)]
-j5 [D0(A5)+D5(A0)]
-j6 [D0(A6)+D6(A0)]
-j7 [D0(A7)+D7(A0)]

[1]
+j0 [D0(A1)+D1(A0)]
-j2 [D1(A2)-D2(A1)] +j3 [D3(A1)-D1(A3)]
-j4 [D1(A4)-D4(A1)] +j5 [D5(A1)-D1(A5)]
-j7 [D1(A7)-D7(A1)] +j6 [D6(A1)-D1(A6)]

[2]
+j0 [D0(A2)+D2(A0)]
-j3 [D2(A3)-D3(A2)] +j1 [D1(A2)-D2(A1)]
-j4 [D2(A4)-D4(A2)] +j6 [D6(A2)-D2(A6)]
-j5 [D2(A5)-D5(A2)] +j7 [D7(A2)-D2(A7)]

[3]
+j0 [D0(A3)+D3(A0)]
-j1 [D3(A1)-D1(A3)] +j2 [D2(A3)-D3(A2)]
-j4 [D3(A4)-D4(A3)] +j7 [D7(A3)-D3(A7)]
-j6 [D3(A6)-D6(A3)] +j5 [D5(A3)-D3(A5)]

[4]
+j0 [D0(A4)+D4(A0)]
-j5 [D4(A5)-D5(A4)] +j1 [D1(A4)-D4(A1)]
-j6 [D4(A6)-D6(A4)] +j2 [D2(A4)-D4(A2)]
-j7 [D4(A7)-D7(A4)] +j3 [D3(A4)-D4(A3)]

[5]
+j0 [D0(A5)+D5(A0)]
-j1 [D5(A1)-D1(A5)] +j4 [D4(A5)-D5(A4)]
-j3 [D5(A3)-D3(A5)] +j6 [D6(A5)-D5(A6)]
-j7 [D5(A7)-D7(A5)] +j2 [D2(A5)-D5(A2)]

[6]
+j0 [D0(A6)+D6(A0)]
-j1 [D6(A1)-D1(A6)] +j7 [D7(A6)-D6(A7)]
-j2 [D6(A2)-D2(A6)] +j4 [D4(A6)-D6(A4)]

-j5 [D6(A5)-D5(A6)] +j3 [D3(A6)-D6(A3)]

[7]
+j0 [D0(A7)+D7(A0)]
-j2 [D7(A2)-D2(A7)] +j5 [D5(A7)-D7(A5)]
-j3 [D7(A3)-D3(A7)] +j4 [D4(A7)-D7(A4)]
-j6 [D7(A6)-D6(A7)] +j1 [D1(A7)-D7(A1)]

Invariant ½ (IL – IR) = 0, IR and IL are irreducible equivalents.

For minimum distance 1: ½[SL(123)+SR(321)]

[4]
-j5 [D7(A6)-D6(A7)] -j6 [D5(A7)-D7(A5)] -j7 [D6(A5)-D5(A6)]

[5]
+j4 [D7(A6)-D6(A7)] +j6 [D4(A7)-D7(A4)] -j7 [D4(A6)-D6(A4)]

[6]
+j4 [D5(A7)-D7(A5)] -j5 [D4(A7)-D7(A4)] +j7 [D4(A5)-D5(A4)]

[7]
+j4 [D6(A5)-D5(A6)] +j5 [D4(A6)-D6(A4)] -j6 [D4(A5)-D5(A4)]

For minimum distance 1: ½[SL(123)-SR(321)]

[0]
-j1 [D2(A3)-D3(A2)] -j2 [D3(A1)-D1(A3)] -j3 [D1(A2)-D2(A1)]

[1]
+j0 [D2(A3)-D3(A2)] -j2 [D0(A3)+D3(A0)] +j3 [D0(A2)+D2(A0)]

[2]
+j0 [D3(A1)-D1(A3)] +j1 [D0(A3)+D3(A0)] -j3 [D0(A1)+D1(A0)]

[3]
+j0 [D1(A2)-D2(A1)] -j1 [D0(A2)+D2(A0)] +j2 [D0(A1)+D1(A0)]

For minimum distance 2: ½[SL(761)+SR(167)]

[2]
-j3 [D4(A5)-D5(A4)] -j4 [D5(A3)-D3(A5)] -j5 [D3(A4)-D4(A3)]

[3]
+j2 [D4(A5)-D5(A4)] -j4 [D2(A5)-D5(A2)] +j5 [D2(A4)-D4(A2)]

[4]
+j2 [D5(A3)-D3(A5)] +j3 [D2(A5)-D5(A2)] -j5 [D2(A3)-D3(A2)]

[5]
+j2 [D3(A4)-D4(A3)] -j3 [D2(A4)-D4(A2)] +j4 [D2(A3)-D3(A2)]

For minimum distance 2: ½[SL(761)-SR(167)]

[0]
-j1 [D7(A6)-D6(A7)] -j6 [D1(A7)-D7(A1)] -j7 [D6(A1)-D1(A6)]

[1]
+j0 [D7(A6)-D6(A7)] +j6 [D0(A7)+D7(A0)] -j7 [D0(A6)+D6(A0)]

[6]
+j0 [D1(A7)-D7(A1)] -j1 [D0(A7)-D7(A0)] +j7 [D0(A1)+D1(A0)]

[7]
+j0 [D6(A1)-D1(A6)] +j1 [D0(A6)+D6(A0)] -j6 [D0(A1)+D1(A0)]

For minimum distance 3: ½[SL(572)+SR(275)]

[1]
+j3 [D4(A6)-D6(A4)] -j4 [D3(A6)-D6(A3)] +j6 [D3(A4)-D4(A3)]

[3]
-j1 [D4(A6)-D6(A4)] -j4 [D6(A1)-D1(A6)] -j6 [D1(A4)-D4(A1)]

[4]
+j1 [D3(A6)-D6(A3)] +j3 [D6(A1)-D1(A6)] -j6 [D3(A1)-D1(A3)]

[6]
-j1 [D3(A4)-D4(A3)] +j3 [D1(A4)-D4(A1)] +j4 [D3(A1)-D1(A3)]

For minimum distance 3: ½[SL(572)-SR(275)]

[0]
-j2 [D5(A7)-D7(A5)] -j5 [D7(A2)-D2(A7)] -j7 [D2(A5)-D5(A2)]

[2]
+j0 [D5(A7)-D7(A5)] -j5 [D0(A7)+D7(A0)] +j7 [D0(A5)+D5(A0)]

[5]
+j0 [D7(A2)-D2(A7)] +j2 [D0(A7)+D7(A0)] -j7 [D0(A2)+D2(A0)]

[7]
+j0 [D2(A5)-D5(A2)] -j2 [D0(A5)+D5(A0)] +j5 [D0(A2)+D2(A0)]

For minimum distance 4: ½[SL(653)+SR(356)]

[1]
-j2 [D4(A7)-D7(A4)] -j4 [D7(A2)-D2(A7)] -j7 [D2(A4)-D4(A2)]

[2]
+j1 [D4(A7)-D7(A4)] -j4 [D1(A7)-D7(A1)] +j7 [D1(A4)-D4(A1)]

[4]
+j1 [D7(A2)-D2(A7)] +j2 [D1(A7)-D7(A1)] -j7 [D1(A2)-D2(A1)]

[7]
+j1 [D2(A4)-D4(A2)] -j2 [D1(A4)-D4(A1)] +j4 [D1(A2)-D2(A1)]

For minimum distance 4: ½[SL(653)-SR(356)]

[0]
-j3 [D6(A5)-D5(A6)] -j5 [D3(A6)-D6(A3)] -j6 [D5(A3)-D3(A5)]

[3]
+j0 [D6(A5)-D5(A6)] +j5 [D0(A6)+D6(A0)] -j6 [D0(A5)+D5(A0)]

[5]
+j0 [D3(A6)-D6(A3)] -j3 [D0(A6)-D6(A0)] +j6 [D0(A3)+D3(A0)]

[6]
+j0 [D5(A3)-D3(A5)] +j3 [D0(A5)+D5(A0)] -j5 [D0(A3)+D3(A0)]

For minimum distance 5: ½[SL(145)+SR(541)]

[2]
-j3 [D7(A6)-D6(A7)] +j6 [D7(A3)-D3(A7)] +j7 [D3(A6)-D6(A3)]

[3]
+j2 [D7(A6)-D6(A7)] -j6 [D7(A2)-D2(A7)] +j7 [D6(A2)-D2(A6)]

[6]
-j2 [D7(A3)-D3(A7)] +j3 [D7(A2)-D2(A7)] +j7 [D2(A3)-D3(A2)]

[7]
-j2 [D3(A6)-D6(A3)] -j3 [D6(A2)-D2(A6)] -j6 [D2(A3)-D3(A2)]

For minimum distance 5: ½[SL(145)-SR(541)]

[0]
-j1 [D4(A5)-D5(A4)] -j4 [D5(A1)-D1(A5)] -j5 [D1(A4)-D4(A1)]

[1]
+j0 [D4(A5)-D5(A4)] -j4 [D0(A5)+D5(A0)] +j5 [D0(A4)+D4(A0)]

[4]
+j0 [D5(A1)-D1(A5)] +j1 [D0(A5)+D5(A0)] -j5 [D0(A1)+D1(A0)]

[5]
+j0 [D1(A4)-D4(A1)] -j1 [D0(A4)+D4(A0)] +j4 [D0(A1)+D1(A0)]

For minimum distance 6: ½[SL(246)+SR(642)]

[1]
+j3 [D5(A7)-D7(A5)] +j5 [D7(A3)-D3(A7)] -j7 [D5(A3)-D3(A5)]

[3]
-j1 [D5(A7)-D7(A5)] +j5 [D1(A7)-D7(A1)] +j7 [D5(A1)-D1(A5)]

[5]
-j1 [D7(A3)-D3(A7)] -j3 [D1(A7)-D7(A1)] -j7 [D3(A1)-D1(A3)]

[7]
+j1 [D5(A3)-D3(A5)] -j3 [D5(A1)-D1(A5)] +j5 [D3(A1)-D1(A3)]

For minimum distance 6: ½[SL(246)-SR(642)]

[0]
-j2 [D4(A6)-D6(A4)] -j4 [D6(A2)-D2(A6)] -j6 [D2(A4)-D4(A2)]

[2]
+j0 [D4(A6)-D6(A4)] -j4 [D0(A6)+D6(A0)] +j6 [D0(A4)+D4(A0)]

[4]

+j0 [D6(A2)-D2(A6)] +j2 [D0(A6)+D6(A0)] -j6 [D0(A2)+D2(A0)]

[6]
+j0 [D2(A4)-D4(A2)] -j2 [D0(A4)+D4(A0)] +j4 [D0(A2)+D2(A0)]

For minimum distance 7: ½[SL(347)+SR(743)]

[1]
-j2 [D6(A5)-D5(A6)] +j5 [D6(A2)-D2(A6)] +j6 [D2(A5)-D5(A2)]

[2]
+j1 [D6(A5)-D5(A6)] -j5 [D6(A1)-D1(A6)] +j6 [D5(A1)-D1(A5)]

[5]
-j1 [D6(A2)-D2(A6)] +j2 [D6(A1)-D1(A6)] +j6 [D1(A2)-D2(A1)]

[6]
-j1 [D2(A5)-D5(A2)] -j2 [D5(A1)-D1(A5)] -j5 [D1(A2)-D2(A1)]

For minimum distance 7: ½[SL(347)-SR(743)]

[0]
-j3 [D4(A7)-D7(A4)] -j4 [D7(A3)-D3(A7)] -j7 [D3(A4)-D4(A3)]

[3]
+j0 [D4(A7)-D7(A4)] -j4 [D0(A7)+D7(A0)] +j7 [D0(A4)+D4(A0)]

[4]
+j0 [D7(A3)-D3(A7)] +j3 [D0(A7)+D7(A0)] -j7 [D0(A3)+D3(A0)]

[7]
+j0 [D3(A4)-D4(A3)] -j3 [D0(A4)+D4(A0)] +j4 [D0(A3)+D3(A0)]

If the full F*j **O** product is to represent the Action Function, then all of the minimum distance functions above must be equated to zero. This would functionally restrict the choices for the potentials. Certainly, if the **O** Action Function presented here is to unify Electrodynamics and Gravitation, there must be potential function choices that produce all experimentally indicated forces, and do not produce forces not indicated by experimental evidence.

Electrodynamics would require the j0 term to cover the charge density. Gravitation would require it to cover the mass density. One would anticipate the two central forces on charge and mass to be the product of a scalar term and an irrotational field. Both are covered in the Action Function above without distinction between mass and charge density, j0 always includes both. The gravitational central force appears in **O** units 1,2 and 3. The central charge force appears in **O** units 5,6 and 7.

If j5, j6 ad j7 are indicative of charge current, then the familiar current – magnetic field vector product force appears in **O** units 5,6 and 7 above. This definition of charge current also produces the expected mechanical work function of Electrodynamics produced by the scalar product of current and electric field.

Each force term in the invariant portion of the Action Function is either an (invariant scalar) * (invariant vector) product or a double **O** product by the rules of a single permutation. In the latter, when an **O** algebra change negates that permutation rule, using the rule twice makes the new algebra result the same as the original algebra result. The scalar work functions are the product of like unit invariants, and these are always invariant. Looking at Algebraic Invariance this way will be very handy when the Action Function is integrated in what follows to form the analogous stress-energy-momentum "tensor", since there is no suitable full **O** product form that can be cranked through the variance sieve process.

Octonion Energy and Momentum Conservation

The next step in the Electrodynamics roadmap is to produce the conservation of momentum and energy equations. This is done by first casting the Action Function in a form that includes an outside differentiation on every term. The unit of the outside differential and the 8 resultant **O** units will be the indices of the Analogous Stress-Energy-Momentum "tensor". The Action Function will then be the divergence of this form. When this representation is integrated over the spatial 7-volume, the outside differentiation can be directly integrated away if also spatial, leaving a 6-surface integral. The non-spatial (time) outside differentials remain volume integrals. The conservation laws express that the sum of the time rate of change in something inside the volume, its flux through its enclosing surface and the Action Function external force and work balance to zero.

So the first step is to express the Action Function in the following form

$$\text{Invariant } F*j = u_i\, \partial/\partial r_i \;\; [(\partial A_k/\partial r_j\;\; u_k\, u_j \;\; + \partial A_j/\partial r_k\;\; u_j\, u_k\,)\, (\partial A_m/\partial r_l\;\; u_l\, u_m \;\; + \partial A_l/\partial r_m\;\; u_m\, u_l\,)]$$

Without some major help, equating this to the invariant portion of F*j would be extremely difficult. Fortunately the Law of Algebraic Invariance gives us all of the necessary clues.

The algebraic character of the new representation of the invariant Action Function is found within four unit products of the form

$$u_i \{ [u_k\, u_j] [u_l\, u_m] \}$$

The desired invariant Action Function values are the full complement of invariance in F*j, so a good guess for its new form is to look for all possible ijklm unit combinations that are algebraic invariants. Clearly there must be no variant combinations included. Invariance is not impacted by commutation of either unit pair inside the square brackets, nor will it be by commutating the two square brackets. This is implied below without showing all juxtapositions. Once invariant combinations of units are in hand, they will be inserted in the field product form above, and the juxtaposition will be adjusted to get the desired results. All possible juxtapositions will not be used, since the natural anti-commutation within **O** algebra would sum for instance $E_x E_y$ and $E_y E_x$ to zero. Electrodynamics tells us that this form is present in the desired result. Anti-commutation is also why a variance sieve on the **O** product square of the field form inside the [] above will not work. All possible combinations would be hit, but the anti-commutation of some **O** product terms would remove many needed terms.

If l=k and m=j, the product [$u_k u_j$] [$u_k u_j$] is a scalar algebraic invariant for all k and j. This indicates the invariance of the index combination

u_i { [$u_k u_j$] [$u_k u_j$] } for any ijk and j not = k (no scalar F): Invariant Form 1

Next, if we let k = l = 0 and j = i , and i not = m, then the result is a new invariant form

u_i { [$u_0 u_i$] [$u_0 u_m$] } for i not = m, and i and m not 0: Invariant Form 2

This is a double application of the permutation rule that includes (im), hence invariant.

If k = 0, and l = j , then [$u_0 u_j$] [$u_j u_m$] is an invariant vector = (+/-)u_m , being a double application of the permutation rule that includes j and m. Therefore, if i = 0 or i = m, the following is also invariant

u_i { [$u_0 u_j$] [$u_j u_m$] } for j not = m or 0, m not = 0, and i = 0 or i = m: Invariant Form 3

Another invariant form would be a quad application of a single permutation rule, say (ijk). This invariant form is represented by

u_j { [$u_k u_j$] [$u_i u_k$] } and u_k { [$u_j u_k$] [$u_i u_j$] } for permutation +(ijk): Invariant Form 4

The fifth and final invariant form is a bit less obvious, and involves a single application of each of the permutation rules from 4 permutations that do not include 1 of the vector units. To see that this indeed is an invariant, remember the rules for negations that yield a valid **O** algebra, either all 3 that include a single unit or all 4 that do not include a single unit. If the unit in question is the unit not in the 4 permutations chosen for our new form for a portion of Invariant(F*j), the non-isomorphic map does not touch any of our choices, and the isomorphic map touches all 4. Both leave our new form invariant.

Next, the 6 units which make up the 4 chosen permutation rules for our last invariant form each appear in 2 of the 4 chosen permutations. So the valid maps of negating all permutations that either include or do not include a single unit in the set of 6 here will always negate 2 of the 4 chosen permutations, and hence leave our new form invariant.

u_i { [$u_i u_j$] [$u_j u_k$] } Invariant Form 5

where the following permutations or their negations are implied, consistent with valid **O** algebra.

(ijl) (jkm) (lmn) (ink)

If we look at the 5 invariant candidate forms, we can identify some of the Electrodynamics stress-energy-momentum tensor forms we need to account for here. Form 1 will cover the time rate of change and gradient of energy density. Form 2 will cover familiar Electrodynamics forms E_xE_y and the like. Form 3 covers the time rate of change and divergence of the Poynting vector, which is itself an algebraic invariant as expected. Form 5 covers forms B_xB_y and the like.

Combining these 5 invariant forms, the following is an equality:

Invariant(F*j) =

[0] m=1-7 (7 terms)
$- \frac{1}{2} u_0 \, \partial/\partial r_0 \; [(\partial A_0/\partial r_m \; u_m u_0 + \partial A_m/\partial r_0 \; u_0 u_m)(\partial A_0/\partial r_m \; u_m u_0 + \partial A_m/\partial r_0 \; u_0 u_m)]$

[0] all permutations +(mnp) (21 terms)
$- \frac{1}{2} u_0 \, \partial/\partial r_0 \; [(\partial A_n/\partial r_m \; u_m u_n + \partial A_m/\partial r_n \; u_n u_m)(\partial A_n/\partial r_m \; u_m u_n + \partial A_m/\partial r_n \; u_n u_m)]$

[0] m = 1-7, n not = 0, m (42 terms)
$+u_m \, \partial/\partial r_m \; [(\partial A_0/\partial r_n \; u_n u_0 + \partial A_n/\partial r_0 \; u_0 u_n)(\partial A_m/\partial r_n \; u_n u_m + \partial A_n/\partial r_m \; u_m u_n)]$

[m] m = 1-7, n not = 0, m (42 terms)
$+u_0 \, \partial/\partial r_0 \; [(\partial A_0/\partial r_n \; u_n u_0 + \partial A_n/\partial r_0 \; u_0 u_n)(\partial A_m/\partial r_n \; u_n u_m + \partial A_n/\partial r_m \; u_m u_n)]$

[m] m = 1-7, n not = 0 or m (42 terms)
$+u_n \, \partial/\partial r_n \; [(\partial A_0/\partial r_m \; u_m u_0 + \partial A_m/\partial r_0 \; u_0 u_m)(\partial A_0/\partial r_n \; u_n u_0 + \partial A_n/\partial r_0 \; u_0 u_n)]$

[m] m = 1-7, all permutations +(mnp) (42 terms)
$+u_n \, \partial/\partial r_n \; [(\partial A_n/\partial r_p \; u_p u_n + \partial A_p/\partial r_n \; u_n u_p)(\partial A_p/\partial r_m \; u_m u_p + \partial A_m/\partial r_p \; u_p u_m)]$
$+u_p \, \partial/\partial r_p \; [(\partial A_p/\partial r_n \; u_n u_p + \partial A_n/\partial r_p \; u_p u_n)(\partial A_n/\partial r_m \; u_m u_n + \partial A_m/\partial r_n \; u_n u_m)]$

[m] m = 1-7, mnp different and not 0, (mnp) not a permutation (168 terms)
$+u_n \, \partial/\partial r_n \; [(\partial A_p/\partial r_m \; u_m u_p + \partial A_m/\partial r_p \; u_p u_m)(\partial A_n/\partial r_p \; u_p u_n + \partial A_p/\partial r_n \; u_n u_p)]$

[m] m = 1-7, n not = 0, m (42 terms)
$+ \frac{1}{2} u_m \, \partial/\partial r_m \; [(\partial A_0/\partial r_n \; u_n u_0 + \partial A_n/\partial r_0 \; u_0 u_n)(\partial A_0/\partial r_n \; u_n u_0 + \partial A_n/\partial r_0 \; u_0 u_n)]$

[m] m = 1-7, (7 terms)
$- \frac{1}{2} u_m \, \partial/\partial r_m \; [(\partial A_0/\partial r_m \; u_m u_0 + \partial A_m/\partial r_0 \; u_0 u_m)(\partial A_0/\partial r_m \; u_m u_0 + \partial A_m/\partial r_0 \; u_0 u_m)]$

[m] m = 1-7, q = 1-7 all +(npq) with n and p not = m, (105 terms)
$- \frac{1}{2} u_m \, \partial/\partial r_m \; [(\partial A_p/\partial r_n \; u_n u_p + \partial A_n/\partial r_p \; u_p u_n)(\partial A_p/\partial r_n \; u_n u_p + \partial A_n/\partial r_p \; u_p u_n)]$

[m] m = 1-7, n not = 0 or m (42 terms)
$+ \frac{1}{2} u_m \partial/\partial r_m \; [(\partial A_n/\partial r_m \; u_m u_n + \partial A_m/\partial r_n \; u_n u_m)(\partial A_n/\partial r_m \; u_m u_n + \partial A_m/\partial r_n \; u_n u_m)]$

Running through all of these, the result is the following equality.

The Action Function Invariant(F * j) =

[0]
+ ½ D0 { [D0(A1)+D1(A0)][D0(A1)+D1(A0)] }
+ ½ D0 { [D0(A2)+D2(A0)][D0(A2)+D2(A0)] }
+ ½ D0 { [D0(A3)+D3(A0)][D0(A3)+D3(A0)] }
+ ½ D0 { [D0(A4)+D4(A0)][D0(A4)+D4(A0)] }
+ ½ D0 { [D0(A5)+D5(A0)][D0(A5)+D5(A0)] }
+ ½ D0 { [D0(A6)+D6(A0)][D0(A6)+D6(A0)] }
+ ½ D0 { [D0(A7)+D7(A0)][D0(A7)+D7(A0)] }
+ ½ D0 { [D1(A2)-D2(A1)][D1(A2)-D2(A1)] }
+ ½ D0 { [D2(A3)-D3(A2)][D2(A3)-D3(A2)] }
+ ½ D0 { [D3(A1)-D1(A3)][D3(A1)-D1(A3)] }
+ ½ D0 { [D7(A6)-D6(A7)][D7(A6)-D6(A7)] }
+ ½ D0 { [D6(A1)-D1(A6)][D6(A1)-D1(A6)] }
+ ½ D0 { [D1(A7)-D7(A1)][D1(A7)-D7(A1)] }
+ ½ D0 { [D5(A7)-D7(A5)][D5(A7)-D7(A5)] }
+ ½ D0 { [D7(A2)-D2(A7)][D7(A2)-D2(A7)] }
+ ½ D0 { [D2(A5)-D5(A2)][D2(A5)-D5(A2)] }
+ ½ D0 { [D6(A5)-D5(A6)][D6(A5)-D5(A6)] }
+ ½ D0 { [D5(A3)-D3(A5)][D5(A3)-D3(A5)] }
+ ½ D0 { [D3(A6)-D6(A3)][D3(A6)-D6(A3)] }
+ ½ D0 { [D1(A4)-D4(A1)][D1(A4)-D4(A1)] }
+ ½ D0 { [D4(A5)-D5(A4)][D4(A5)-D5(A4)] }
+ ½ D0 { [D5(A1)-D1(A5)][D5(A1)-D1(A5)] }
+ ½ D0 { [D2(A4)-D4(A2)][D2(A4)-D4(A2)] }
+ ½ D0 { [D4(A6)-D6(A4)][D4(A6)-D6(A4)] }
+ ½ D0 { [D6(A2)-D2(A6)][D6(A2)-D2(A6)] }
+ ½ D0 { [D3(A4)-D4(A3)][D3(A4)-D4(A3)] }
+ ½ D0 { [D4(A7)-D7(A4)][D4(A7)-D7(A4)] }
+ ½ D0 { [D7(A3)-D3(A7)][D7(A3)-D3(A7)] }

+D1 { [D0(A2)+D2(A0)][D2(A1)-D1(A2)] }
+D1 { [D0(A3)+D3(A0)][D3(A1)-D1(A3)] }
+D1 { [D0(A4)+D4(A0)][D4(A1)-D1(A4)] }
+D1 { [D0(A5)+D5(A0)][D5(A1)-D1(A5)] }
+D1 { [D0(A6)+D6(A0)][D6(A1)-D1(A6)] }
+D1 { [D0(A7)+D7(A0)][D7(A1)-D1(A7)] }

+D2 { [D0(A1)+D1(A0)][D1(A2)-D2(A1)] }
+D2 { [D0(A3)+D3(A0)][D3(A2)-D2(A3)] }

+D2 { [D0(A4)+D4(A0)][D4(A2)-D2(A4)] }
+D2 { [D0(A5)+D5(A0)][D5(A2)-D2(A5)] }
+D2 { [D0(A6)+D6(A0)][D6(A2)-D2(A6)] }
+D2 { [D0(A7)+D7(A0)][D7(A2)-D2(A7)] }

+D3 { [D0(A1)+D1(A0)][D1(A3)-D3(A1)] }
+D3 { [D0(A2)+D2(A0)][D2(A3)-D3(A2)] }
+D3 { [D0(A4)+D4(A0)][D4(A3)-D3(A4)] }
+D3 { [D0(A5)+D5(A0)][D5(A3)-D3(A5)] }
+D3 { [D0(A6)+D6(A0)][D6(A3)-D3(A6)] }
+D3 { [D0(A7)+D7(A0)][D7(A3)-D3(A7)] }

+D4 { [D0(A1)+D1(A0)][D1(A4)-D4(A1)] }
+D4 { [D0(A2)+D2(A0)][D2(A4)-D4(A2)] }
+D4 { [D0(A3)+D3(A0)][D3(A4)-D4(A3)] }
+D4 { [D0(A5)+D5(A0)][D5(A4)-D4(A5)] }
+D4 { [D0(A6)+D6(A0)][D6(A4)-D4(A6)] }
+D4 { [D0(A7)+D7(A0)][D7(A4)-D4(A7)] }

+D5 { [D0(A1)+D1(A0)][D1(A5)-D5(A1)] }
+D5 { [D0(A2)+D2(A0)][D2(A5)-D5(A2)] }
+D5 { [D0(A3)+D3(A0)][D3(A5)-D5(A3)] }
+D5 { [D0(A4)+D4(A0)][D4(A5)-D5(A4)] }
+D5 { [D0(A6)+D6(A0)][D6(A5)-D5(A6)] }
+D5 { [D0(A7)+D7(A0)][D7(A5)-D5(A7)] }

+D6 { [D0(A1)+D1(A0)][D1(A6)-D6(A1)] }
+D6 { [D0(A2)+D2(A0)][D2(A6)-D6(A2)] }
+D6 { [D0(A3)+D3(A0)][D3(A6)-D6(A3)] }
+D6 { [D0(A4)+D4(A0)][D4(A6)-D6(A4)] }
+D6 { [D0(A5)+D5(A0)][D5(A6)-D6(A5)] }
+D6 { [D0(A7)+D7(A0)][D7(A6)-D6(A7)] }

+D7 { [D0(A1)+D1(A0)][D1(A7)-D7(A1)] }
+D7 { [D0(A2)+D2(A0)][D2(A7)-D7(A2)] }
+D7 { [D0(A3)+D3(A0)][D3(A7)-D7(A3)] }
+D7 { [D0(A4)+D4(A0)][D4(A7)-D7(A4)] }
+D7 { [D0(A5)+D5(A0)][D5(A7)-D7(A5)] }
+D7 { [D0(A6)+D6(A0)][D6(A7)-D7(A6)] }

[1]
-D0 { [D0(A2)+D2(A0)][D2(A1)-D1(A2)] }
-D0 { [D0(A3)+D3(A0)][D3(A1)-D1(A3)] }
-D0 { [D0(A4)+D4(A0)][D4(A1)-D1(A4)] }
-D0 { [D0(A5)+D5(A0)][D5(A1)-D1(A5)] }

-D0 { [D0(A6)+D6(A0)][D6(A1)-D1(A6)] }
-D0 { [D0(A7)+D7(A0)][D7(A1)-D1(A7)] }

+ ½ D1 { [D0(A1)+D1(A0)][D0(A1)+D1(A0)] }
- ½ D1 { [D0(A2)+D2(A0)][D0(A2)+D2(A0)] }
- ½ D1 { [D0(A3)+D3(A0)][D0(A3)+D3(A0)] }
- ½ D1 { [D0(A4)+D4(A0)][D0(A4)+D4(A0)] }
- ½ D1 { [D0(A5)+D5(A0)][D0(A5)+D5(A0)] }
- ½ D1 { [D0(A6)+D6(A0)][D0(A6)+D6(A0)] }
- ½ D1 { [D0(A7)+D7(A0)][D0(A7)+D7(A0)] }
- ½ D1 { [D1(A2)-D2(A1)][D1(A2)-D2(A1)] }
+ ½ D1 { [D2(A3)-D3(A2)][D2(A3)-D3(A2)] }
- ½ D1 { [D3(A1)-D1(A3)][D3(A1)-D1(A3)] }
+ ½ D1 { [D7(A6)-D6(A7)][D7(A6)-D6(A7)] }
- ½ D1 { [D6(A1)-D1(A6)][D6(A1)-D1(A6)] }
- ½ D1 { [D1(A7)-D7(A1)][D1(A7)-D7(A1)] }
+ ½ D1 { [D5(A7)-D7(A5)][D5(A7)-D7(A5)] }
+ ½ D1 { [D7(A2)-D2(A7)][D7(A2)-D2(A7)] }
+ ½ D1 { [D2(A5)-D5(A2)][D2(A5)-D5(A2)] }
+ ½ D1 { [D6(A5)-D5(A6)][D6(A5)-D5(A6)] }
+ ½ D1 { [D5(A3)-D3(A5)][D5(A3)-D3(A5)] }
+ ½ D1 { [D3(A6)-D6(A3)][D3(A6)-D6(A3)] }
- ½ D1 { [D1(A4)-D4(A1)][D1(A4)-D4(A1)] }
+ ½ D1 { [D4(A5)-D5(A4)][D4(A5)-D5(A4)] }
- ½ D1 { [D5(A1)-D1(A5)][D5(A1)-D1(A5)] }
+ ½ D1 { [D2(A4)-D4(A2)][D2(A4)-D4(A2)] }
+ ½ D1 { [D4(A6)-D6(A4)][D4(A6)-D6(A4)] }
+ ½ D1 { [D6(A2)-D2(A6)][D6(A2)-D2(A6)] }
+ ½ D1 { [D3(A4)-D4(A3)][D3(A4)-D4(A3)] }
+ ½ D1 { [D4(A7)-D7(A4)][D4(A7)-D7(A4)] }
+ ½ D1 { [D7(A3)-D3(A7)][D7(A3)-D3(A7)] }

+D2 { [D0(A1)+D1(A0)][D0(A2)+D2(A0)] }
+D2 { [D3(A2)-D2(A3)][D1(A3)-D3(A1)] }
+D2 { [D4(A2)-D2(A4)][D1(A4)-D4(A1)] }
+D2 { [D5(A2)-D2(A5)][D1(A5)-D5(A1)] }
+D2 { [D6(A2)-D2(A6)][D1(A6)-D6(A1)] }
+D2 { [D7(A2)-D2(A7)][D1(A7)-D7(A1)] }

+D3 { [D0(A1)+D1(A0)][D0(A3)+D3(A0)] }
+D3 { [D2(A3)-D3(A2)][D1(A2)-D2(A1)] }
+D3 { [D4(A3)-D3(A4)][D1(A4)-D4(A1)] }
+D3 { [D5(A3)-D3(A5)][D1(A5)-D5(A1)] }
+D3 { [D6(A3)-D3(A6)][D1(A6)-D6(A1)] }
+D3 { [D7(A3)-D3(A7)][D1(A7)-D7(A1)] }

+D4 { [D0(A1)+D1(A0)][D0(A4)+D4(A0)] }
+D4 { [D2(A4)-D4(A2)][D1(A2)-D2(A1)] }
+D4 { [D3(A4)-D4(A3)][D1(A3)-D3(A1)] }
+D4 { [D5(A4)-D4(A5)][D1(A5)-D5(A1)] }
+D4 { [D6(A4)-D4(A6)][D1(A6)-D6(A1)] }
+D4 { [D7(A4)-D4(A7)][D1(A7)-D7(A1)] }

+D5 { [D0(A1)+D1(A0)][D0(A5)+D5(A0)] }
+D5 { [D2(A5)-D5(A2)][D1(A2)-D2(A1)] }
+D5 { [D3(A5)-D5(A3)][D1(A3)-D3(A1)] }
+D5 { [D4(A5)-D5(A4)][D1(A4)-D4(A1)] }
+D5 { [D6(A5)-D5(A6)][D1(A6)-D6(A1)] }
+D5 { [D7(A5)-D5(A7)][D1(A7)-D7(A1)] }

+D6 { [D0(A1)+D1(A0)][D0(A6)+D6(A0)] }
+D6 { [D2(A6)-D6(A2)][D1(A2)-D2(A1)] }
+D6 { [D3(A6)-D6(A3)][D1(A3)-D3(A1)] }
+D6 { [D4(A6)-D6(A4)][D1(A4)-D4(A1)] }
+D6 { [D5(A6)-D6(A5)][D1(A5)-D5(A1)] }
+D6 { [D7(A6)-D6(A7)][D1(A7)-D7(A1)] }

+D7 { [D0(A1)+D1(A0)][D0(A7)+D7(A0)] }
+D7 { [D2(A7)-D7(A2)][D1(A2)-D2(A1)] }
+D7 { [D3(A7)-D7(A3)][D1(A3)-D3(A1)] }
+D7 { [D4(A7)-D7(A4)][D1(A4)-D4(A1)] }
+D7 { [D5(A7)-D7(A5)][D1(A5)-D5(A1)] }
+D7 { [D6(A7)-D7(A6)][D1(A6)-D6(A1)] }

[2]
-D0 { [D0(A1)+D1(A0)][D1(A2)-D2(A1)] }
-D0 { [D0(A3)+D3(A0)][D3(A2)-D2(A3)] }
-D0 { [D0(A4)+D4(A0)][D4(A2)-D2(A4)] }
-D0 { [D0(A5)+D5(A0)][D5(A2)-D2(A5)] }
-D0 { [D0(A6)+D6(A0)][D6(A2)-D2(A6)] }
-D0 { [D0(A7)+D7(A0)][D7(A2)-D2(A7)] }

+D1 { [D0(A2)+D2(A0)][D0(A1)+D1(A0)] }
+D1 { [D3(A1)-D1(A3)][D2(A3)-D3(A2)] }
+D1 { [D4(A1)-D1(A4)][D2(A4)-D4(A2)] }
+D1 { [D5(A1)-D1(A5)][D2(A5)-D5(A2)] }
+D1 { [D6(A1)-D1(A6)][D2(A6)-D6(A2)] }
+D1 { [D7(A1)-D1(A7)][D2(A7)-D7(A2)] }

- ½ D2 { [D0(A1)+D1(A0)][D0(A1)+D1(A0)] }
+ ½ D2 { [D0(A2)+D2(A0)][D0(A2)+D2(A0)] }

- ½ D2 { [D0(A3)+D3(A0)][D0(A3)+D3(A0)] }
- ½ D2 { [D0(A4)+D4(A0)][D0(A4)+D4(A0)] }
- ½ D2 { [D0(A5)+D5(A0)][D0(A5)+D5(A0)] }
- ½ D2 { [D0(A6)+D6(A0)][D0(A6)+D6(A0)] }
- ½ D2 { [D0(A7)+D7(A0)][D0(A7)+D7(A0)] }
- ½ D2 { [D1(A2)-D2(A1)][D1(A2)-D2(A1)] }
- ½ D2 { [D2(A3)-D3(A2)][D2(A3)-D3(A2)] }
+ ½ D2 { [D3(A1)-D1(A3)][D3(A1)-D1(A3)] }
+ ½ D2 { [D7(A6)-D6(A7)][D7(A6)-D6(A7)] }
+ ½ D2 { [D6(A1)-D1(A6)][D6(A1)-D1(A6)] }
+ ½ D2 { [D1(A7)-D7(A1)][D1(A7)-D7(A1)] }
+ ½ D2 { [D5(A7)-D7(A5)][D5(A7)-D7(A5)] }
- ½ D2 { [D7(A2)-D2(A7)][D7(A2)-D2(A7)] }
- ½ D2 { [D2(A5)-D5(A2)][D2(A5)-D5(A2)] }
+ ½ D2 { [D6(A5)-D5(A6)][D6(A5)-D5(A6)] }
+ ½ D2 { [D5(A3)-D3(A5)][D5(A3)-D3(A5)] }
+ ½ D2 { [D3(A6)-D6(A3)][D3(A6)-D6(A3)] }
+ ½ D2 { [D1(A4)-D4(A1)][D1(A4)-D4(A1)] }
+ ½ D2 { [D4(A5)-D5(A4)][D4(A5)-D5(A4)] }
+ ½ D2 { [D5(A1)-D1(A5)][D5(A1)-D1(A5)] }
- ½ D2 { [D2(A4)-D4(A2)][D2(A4)-D4(A2)] }
+ ½ D2 { [D4(A6)-D6(A4)][D4(A6)-D6(A4)] }
- ½ D2 { [D6(A2)-D2(A6)][D6(A2)-D2(A6)] }
+ ½ D2 { [D3(A4)-D4(A3)][D3(A4)-D4(A3)] }
+ ½ D2 { [D4(A7)-D7(A4)][D4(A7)-D7(A4)] }
+ ½ D2 { [D7(A3)-D3(A7)][D7(A3)-D3(A7)] }

+D3 { [D0(A2)+D2(A0)][D0(A3)+D3(A0)] }
+D3 { [D1(A3)-D3(A1)][D2(A1)-D1(A2)] }
+D3 { [D4(A3)-D3(A4)][D2(A4)-D4(A2)] }
+D3 { [D5(A3)-D3(A5)][D2(A5)-D5(A2)] }
+D3 { [D6(A3)-D3(A6)][D2(A6)-D6(A2)] }
+D3 { [D7(A3)-D3(A7)][D2(A7)-D7(A2)] }

+D4 { [D0(A2)+D2(A0)][D0(A4)+D4(A0)] }
+D4 { [D1(A4)-D4(A1)][D2(A1)-D1(A2)] }
+D4 { [D3(A4)-D4(A3)][D2(A3)-D3(A2)] }
+D4 { [D5(A4)-D4(A5)][D2(A5)-D5(A2)] }
+D4 { [D6(A4)-D4(A6)][D2(A6)-D6(A2)] }
+D4 { [D7(A4)-D4(A7)][D2(A7)-D7(A2)] }

+D5 { [D0(A2)+D2(A0)][D0(A5)+D5(A0)] }
+D5 { [D1(A5)-D5(A1)][D2(A1)-D1(A2)] }
+D5 { [D3(A5)-D5(A3)][D2(A3)-D3(A2)] }
+D5 { [D4(A5)-D5(A4)][D2(A4)-D4(A2)] }
+D5 { [D6(A5)-D5(A6)][D2(A6)-D6(A2)] }

+D5 { [D7(A5)-D5(A7)][D2(A7)-D7(A2)] }

+D6 { [D0(A2)+D2(A0)][D0(A6)+D6(A0)] }
+D6 { [D1(A6)-D6(A1)][D2(A1)-D1(A2)] }
+D6 { [D3(A6)-D6(A3)][D2(A3)-D3(A2)] }
+D6 { [D4(A6)-D6(A4)][D2(A4)-D4(A2)] }
+D6 { [D5(A6)-D6(A5)][D2(A5)-D5(A2)] }
+D6 { [D7(A6)-D6(A7)][D2(A7)-D7(A2)] }

+D7 { [D0(A2)+D2(A0)][D0(A7)+D7(A0)] }
+D7 { [D1(A7)-D7(A1)][D2(A1)-D1(A2)] }
+D7 { [D3(A7)-D7(A3)][D2(A3)-D3(A2)] }
+D7 { [D4(A7)-D7(A4)][D2(A4)-D4(A2)] }
+D7 { [D5(A7)-D7(A5)][D2(A5)-D5(A2)] }
+D7 { [D6(A7)-D7(A6)][D2(A6)-D6(A2)] }

[3]
-D0 { [D0(A1)+D1(A0)][D1(A3)-D3(A1)] }
-D0 { [D0(A2)+D2(A0)][D2(A3)-D3(A2)] }
-D0 { [D0(A4)+D4(A0)][D4(A3)-D3(A4)] }
-D0 { [D0(A5)+D5(A0)][D5(A3)-D3(A5)] }
-D0 { [D0(A6)+D6(A0)][D6(A3)-D3(A6)] }
-D0 { [D0(A7)+D7(A0)][D7(A3)-D3(A7)] }

+D1 { [D0(A3)+D3(A0)][D0(A1)+D1(A0)] }
+D1 { [D2(A1)-D1(A2)][D3(A2)-D2(A3)] }
+D1 { [D4(A1)-D1(A4)][D3(A4)-D4(A3)] }
+D1 { [D5(A1)-D1(A5)][D3(A5)-D5(A3)] }
+D1 { [D6(A1)-D1(A6)][D3(A6)-D6(A3)] }
+D1 { [D7(A1)-D1(A7)][D3(A7)-D7(A3)] }

+D2 { [D0(A3)+D3(A0)][D0(A2)+D2(A0)] }
+D2 { [D1(A2)-D2(A1)][D3(A1)-D1(A3)] }
+D2 { [D4(A2)-D2(A4)][D3(A4)-D4(A3)] }
+D2 { [D5(A2)-D2(A5)][D3(A5)-D5(A3)] }
+D2 { [D6(A2)-D2(A6)][D3(A6)-D6(A3)] }
+D2 { [D7(A2)-D2(A7)][D3(A7)-D7(A3)] }

- ½ D3 { [D0(A1)+D1(A0)][D0(A1)+D1(A0)] }
- ½ D3 { [D0(A2)+D2(A0)][D0(A2)+D2(A0)] }
+ ½ D3 { [D0(A3)+D3(A0)][D0(A3)+D3(A0)] }
- ½ D3 { [D0(A4)+D4(A0)][D0(A4)+D4(A0)] }
- ½ D3 { [D0(A5)+D5(A0)][D0(A5)+D5(A0)] }
- ½ D3 { [D0(A6)+D6(A0)][D0(A6)+D6(A0)] }
- ½ D3 { [D0(A7)+D7(A0)][D0(A7)+D7(A0)] }

+ ½ D3 { [D1(A2)-D2(A1)][D1(A2)-D2(A1)] }
- ½ D3 { [D2(A3)-D3(A2)][D2(A3)-D3(A2)] }
- ½ D3 { [D3(A1)-D1(A3)][D3(A1)-D1(A3)] }
+ ½ D3 { [D7(A6)-D6(A7)][D7(A6)-D6(A7)] }
+ ½ D3 { [D6(A1)-D1(A6)][D6(A1)-D1(A6)] }
+ ½ D3 { [D1(A7)-D7(A1)][D1(A7)-D7(A1)] }
+ ½ D3 { [D5(A7)-D7(A5)][D5(A7)-D7(A5)] }
+ ½ D3 { [D7(A2)-D2(A7)][D7(A2)-D2(A7)] }
+ ½ D3 { [D2(A5)-D5(A2)][D2(A5)-D5(A2)] }
+ ½ D3 { [D6(A5)-D5(A6)][D6(A5)-D5(A6)] }
- ½ D3 { [D5(A3)-D3(A5)][D5(A3)-D3(A5)] }
- ½ D3 { [D3(A6)-D6(A3)][D3(A6)-D6(A3)] }
+ ½ D3 { [D1(A4)-D4(A1)][D1(A4)-D4(A1)] }
+ ½ D3 { [D4(A5)-D5(A4)][D4(A5)-D5(A4)] }
+ ½ D3 { [D5(A1)-D1(A5)][D5(A1)-D1(A5)] }
+ ½ D3 { [D2(A4)-D4(A2)][D2(A4)-D4(A2)] }
+ ½ D3 { [D4(A6)-D6(A4)][D4(A6)-D6(A4)] }
+ ½ D3 { [D6(A2)-D2(A6)][D6(A2)-D2(A6)] }
- ½ D3 { [D3(A4)-D4(A3)][D3(A4)-D4(A3)] }
+ ½ D3 { [D4(A7)-D7(A4)][D4(A7)-D7(A4)] }
- ½ D3 { [D7(A3)-D3(A7)][D7(A3)-D3(A7)] }

+D4 { [D0(A3)+D3(A0)][D0(A4)+D4(A0)] }
+D4 { [D1(A4)-D4(A1)][D3(A1)-D1(A3)] }
+D4 { [D2(A4)-D4(A2)][D3(A2)-D2(A3)] }
+D4 { [D5(A4)-D4(A5)][D3(A5)-D5(A3)] }
+D4 { [D6(A4)-D4(A6)][D3(A6)-D6(A3)] }
+D4 { [D7(A4)-D4(A7)][D3(A7)-D7(A3)] }

+D5 { [D0(A3)+D3(A0)][D0(A5)+D5(A0)] }
+D5 { [D1(A5)-D5(A1)][D3(A1)-D1(A3)] }
+D5 { [D2(A5)-D5(A2)][D3(A2)-D2(A3)] }
+D5 { [D4(A5)-D5(A4)][D3(A4)-D4(A3)] }
+D5 { [D6(A5)-D5(A6)][D3(A6)-D6(A3)] }
+D5 { [D7(A5)-D5(A7)][D3(A7)-D7(A3)] }

+D6 { [D0(A3)+D3(A0)][D0(A6)+D6(A0)] }
+D6 { [D1(A6)-D6(A1)][D3(A1)-D1(A3)] }
+D6 { [D2(A6)-D6(A2)][D3(A2)-D2(A3)] }
+D6 { [D4(A6)-D6(A4)][D3(A4)-D4(A3)] }
+D6 { [D5(A6)-D6(A5)][D3(A5)-D5(A3)] }
+D6 { [D7(A6)-D6(A7)][D3(A7)-D7(A3)] }

+D7 { [D0(A3)+D3(A0)][D0(A7)+D7(A0)] }
+D7 { [D1(A7)-D7(A1)][D3(A1)-D1(A3)] }
+D7 { [D2(A7)-D7(A2)][D3(A2)-D2(A3)] }

+D7 { [D4(A7)-D7(A4)][D3(A4)-D4(A3)] }
+D7 { [D5(A7)-D7(A5)][D3(A5)-D5(A3)] }
+D7 { [D6(A7)-D7(A6)][D3(A6)-D6(A3)] }

[4]
-D0 { [D0(A1)+D1(A0)][D1(A4)-D4(A1)] }
-D0 { [D0(A2)+D2(A0)][D2(A4)-D4(A2)] }
-D0 { [D0(A3)+D3(A0)][D3(A4)-D4(A3)] }
-D0 { [D0(A5)+D5(A0)][D5(A4)-D4(A5)] }
-D0 { [D0(A6)+D6(A0)][D6(A4)-D4(A6)] }
-D0 { [D0(A7)+D7(A0)][D7(A4)-D4(A7)] }

+D1 { [D0(A4)+D4(A0)][D0(A1)+D1(A0)] }
+D1 { [D2(A1)-D1(A2)][D4(A2)-D2(A4)] }
+D1 { [D3(A1)-D1(A3)][D4(A3)-D3(A4)] }
+D1 { [D5(A1)-D1(A5)][D4(A5)-D5(A4)] }
+D1 { [D6(A1)-D1(A6)][D4(A6)-D6(A4)] }
+D1 { [D7(A1)-D1(A7)][D4(A7)-D7(A4)] }

+D2 { [D0(A4)+D4(A0)][D0(A2)+D2(A0)] }
+D2 { [D1(A2)-D2(A1)][D4(A1)-D1(A4)] }
+D2 { [D3(A2)-D2(A3)][D4(A3)-D3(A4)] }
+D2 { [D5(A2)-D2(A5)][D4(A5)-D5(A4)] }
+D2 { [D6(A2)-D2(A6)][D4(A6)-D6(A4)] }
+D2 { [D7(A2)-D2(A7)][D4(A7)-D7(A4)] }

+D3 { [D0(A4)+D4(A0)][D0(A3)+D3(A0)] }
+D3 { [D1(A3)-D3(A1)][D4(A1)-D1(A4)] }
+D3 { [D2(A3)-D3(A2)][D4(A2)-D2(A4)] }
+D3 { [D5(A3)-D3(A5)][D4(A5)-D5(A4)] }
+D3 { [D6(A3)-D3(A6)][D4(A6)-D6(A4)] }
+D3 { [D7(A3)-D3(A7)][D4(A7)-D7(A4)] }

- ½ D4 { [D0(A1)+D1(A0)][D0(A1)+D1(A0)] }
- ½ D4 { [D0(A2)+D2(A0)][D0(A2)+D2(A0)] }
- ½ D4 { [D0(A3)+D3(A0)][D0(A3)+D3(A0)] }
+ ½ D4 { [D0(A4)+D4(A0)][D0(A4)+D4(A0)] }
- ½ D4 { [D0(A5)+D5(A0)][D0(A5)+D5(A0)] }
- ½ D4 { [D0(A6)+D6(A0)][D0(A6)+D6(A0)] }
- ½ D4 { [D0(A7)+D7(A0)][D0(A7)+D7(A0)] }
+ ½ D4 { [D1(A2)-D2(A1)][D1(A2)-D2(A1)] }
+ ½ D4 { [D2(A3)-D3(A2)][D2(A3)-D3(A2)] }
+ ½ D4 { [D3(A1)-D1(A3)][D3(A1)-D1(A3)] }
+ ½ D4 { [D7(A6)-D6(A7)][D7(A6)-D6(A7)] }
+ ½ D4 { [D6(A1)-D1(A6)][D6(A1)-D1(A6)] }

+ ½ D4 { [D1(A7)-D7(A1)][D1(A7)-D7(A1)] }
+ ½ D4 { [D5(A7)-D7(A5)][D5(A7)-D7(A5)] }
+ ½ D4 { [D7(A2)-D2(A7)][D7(A2)-D2(A7)] }
+ ½ D4 { [D2(A5)-D5(A2)][D2(A5)-D5(A2)] }
+ ½ D4 { [D6(A5)-D5(A6)][D6(A5)-D5(A6)] }
+ ½ D4 { [D5(A3)-D3(A5)][D5(A3)-D3(A5)] }
+ ½ D4 { [D3(A6)-D6(A3)][D3(A6)-D6(A3)] }
- ½ D4 { [D1(A4)-D4(A1)][D1(A4)-D4(A1)] }
- ½ D4 { [D4(A5)-D5(A4)][D4(A5)-D5(A4)] }
+ ½ D4 { [D5(A1)-D1(A5)][D5(A1)-D1(A5)] }
- ½ D4 { [D2(A4)-D4(A2)][D2(A4)-D4(A2)] }
- ½ D4 { [D4(A6)-D6(A4)][D4(A6)-D6(A4)] }
+ ½ D4 { [D6(A2)-D2(A6)][D6(A2)-D2(A6)] }
- ½ D4 { [D3(A4)-D4(A3)][D3(A4)-D4(A3)] }
- ½ D4 { [D4(A7)-D7(A4)][D4(A7)-D7(A4)] }
+ ½ D4 { [D7(A3)-D3(A7)][D7(A3)-D3(A7)] }

+D5 { [D0(A4)+D4(A0)][D0(A5)+D5(A0)] }
+D5 { [D1(A5)-D5(A1)][D4(A1)-D1(A4)] }
+D5 { [D2(A5)-D5(A2)][D4(A2)-D2(A4)] }
+D5 { [D3(A5)-D5(A3)][D4(A3)-D3(A4)] }
+D5 { [D6(A5)-D5(A6)][D4(A6)-D6(A4)] }
+D5 { [D7(A5)-D5(A7)][D4(A7)-D7(A4)] }

+D6 { [D0(A4)+D4(A0)][D0(A6)+D6(A0)] }
+D6 { [D1(A6)-D6(A1)][D4(A1)-D1(A4)] }
+D6 { [D2(A6)-D6(A2)][D4(A2)-D2(A4)] }
+D6 { [D3(A6)-D6(A3)][D4(A3)-D3(A4)] }
+D6 { [D5(A6)-D6(A5)][D4(A5)-D5(A4)] }
+D6 { [D7(A6)-D6(A7)][D4(A7)-D7(A4)] }

+D7 { [D0(A4)+D4(A0)][D0(A7)+D7(A0)] }
+D7 { [D1(A7)-D7(A1)][D4(A1)-D1(A4)] }
+D7 { [D2(A7)-D7(A2)][D4(A2)-D2(A4)] }
+D7 { [D3(A7)-D7(A3)][D4(A3)-D3(A4)] }
+D7 { [D5(A7)-D7(A5)][D4(A5)-D5(A4)] }
+D7 { [D6(A7)-D7(A6)][D4(A6)-D6(A4)] }

[5]
-D0 { [D0(A1)+D1(A0)][D1(A5)-D5(A1)] }
-D0 { [D0(A2)+D2(A0)][D2(A5)-D5(A2)] }
-D0 { [D0(A3)+D3(A0)][D3(A5)-D5(A3)] }
-D0 { [D0(A4)+D4(A0)][D4(A5)-D5(A4)] }
-D0 { [D0(A6)+D6(A0)][D6(A5)-D5(A6)] }
-D0 { [D0(A7)+D7(A0)][D7(A5)-D5(A7)] }

+D1 { [D0(A5)+D5(A0)][D0(A1)+D1(A0)] }
+D1 { [D2(A1)-D1(A2)][D5(A2)-D2(A5)] }
+D1 { [D3(A1)-D1(A3)][D5(A3)-D3(A5)] }
+D1 { [D4(A1)-D1(A4)][D5(A4)-D4(A5)] }
+D1 { [D6(A1)-D1(A6)][D5(A6)-D6(A5)] }
+D1 { [D7(A1)-D1(A7)][D5(A7)-D7(A5)] }

+D2 { [D0(A5)+D5(A0)][D0(A2)+D2(A0)] }
+D2 { [D1(A2)-D2(A1)][D5(A1)-D1(A5)] }
+D2 { [D3(A2)-D2(A3)][D5(A3)-D3(A5)] }
+D2 { [D4(A2)-D2(A4)][D5(A4)-D4(A5)] }
+D2 { [D6(A2)-D2(A6)][D5(A6)-D6(A5)] }
+D2 { [D7(A2)-D2(A7)][D5(A7)-D7(A5)] }

+D3 { [D0(A5)+D5(A0)][D0(A3)+D3(A0)] }
+D3 { [D1(A3)-D3(A1)][D5(A1)-D1(A5)] }
+D3 { [D2(A3)-D3(A2)][D5(A2)-D2(A5)] }
+D3 { [D4(A3)-D3(A4)][D5(A4)-D4(A5)] }
+D3 { [D6(A3)-D3(A6)][D5(A6)-D6(A5)] }
+D3 { [D7(A3)-D3(A7)][D5(A7)-D7(A5)] }

+D4 { [D0(A5)+D5(A0)][D0(A4)+D4(A0)] }
+D4 { [D1(A4)-D4(A1)][D5(A1)-D1(A5)] }
+D4 { [D2(A4)-D4(A2)][D5(A2)-D2(A5)] }
+D4 { [D3(A4)-D4(A3)][D5(A3)-D3(A5)] }
+D4 { [D6(A4)-D4(A6)][D5(A6)-D6(A5)] }
+D4 { [D7(A4)-D4(A7)][D5(A7)-D7(A5)] }

- ½ D5 { [D0(A1)+D1(A0)][D0(A1)+D1(A0)] }
- ½ D5 { [D0(A2)+D2(A0)][D0(A2)+D2(A0)] }
- ½ D5 { [D0(A3)+D3(A0)][D0(A3)+D3(A0)] }
- ½ D5 { [D0(A4)+D4(A0)][D0(A4)+D4(A0)] }
+ ½ D5 { [D0(A5)+D5(A0)][D0(A5)+D5(A0)] }
- ½ D5 { [D0(A6)+D6(A0)][D0(A6)+D6(A0)] }
- ½ D5 { [D0(A7)+D7(A0)][D0(A7)+D7(A0)] }
+ ½ D5 { [D1(A2)-D2(A1)][D1(A2)-D2(A1)] }
+ ½ D5 { [D2(A3)-D3(A2)][D2(A3)-D3(A2)] }
+ ½ D5 { [D3(A1)-D1(A3)][D3(A1)-D1(A3)] }
+ ½ D5 { [D7(A6)-D6(A7)][D7(A6)-D6(A7)] }
+ ½ D5 { [D6(A1)-D1(A6)][D6(A1)-D1(A6)] }
+ ½ D5 { [D1(A7)-D7(A1)][D1(A7)-D7(A1)] }
- ½ D5 { [D5(A7)-D7(A5)][D5(A7)-D7(A5)] }
+ ½ D5 { [D7(A2)-D2(A7)][D7(A2)-D2(A7)] }
- ½ D5 { [D2(A5)-D5(A2)][D2(A5)-D5(A2)] }
- ½ D5 { [D6(A5)-D5(A6)][D6(A5)-D5(A6)] }

- ½ D5 { [D5(A3)-D3(A5)][D5(A3)-D3(A5)] }
+ ½ D5 { [D3(A6)-D6(A3)][D3(A6)-D6(A3)] }
+ ½ D5 { [D1(A4)-D4(A1)][D1(A4)-D4(A1)] }
- ½ D5 { [D4(A5)-D5(A4)][D4(A5)-D5(A4)] }
- ½ D5 { [D5(A1)-D1(A5)][D5(A1)-D1(A5)] }
+ ½ D5 { [D2(A4)-D4(A2)][D2(A4)-D4(A2)] }
+ ½ D5 { [D4(A6)-D6(A4)][D4(A6)-D6(A4)] }
+ ½ D5 { [D6(A2)-D2(A6)][D6(A2)-D2(A6)] }
+ ½ D5 { [D3(A4)-D4(A3)][D3(A4)-D4(A3)] }
+ ½ D5 { [D4(A7)-D7(A4)][D4(A7)-D7(A4)] }
+ ½ D5 { [D7(A3)-D3(A7)][D7(A3)-D3(A7)] }

+D6 { [D0(A5)+D5(A0)][D0(A6)+D6(A0)] }
+D6 { [D1(A6)-D6(A1)][D5(A1)-D1(A5)] }
+D6 { [D2(A6)-D6(A2)][D5(A2)-D2(A5)] }
+D6 { [D3(A6)-D6(A3)][D5(A3)-D3(A5)] }
+D6 { [D4(A6)-D6(A4)][D5(A4)-D4(A5)] }
+D6 { [D7(A6)-D6(A7)][D5(A7)-D7(A5)] }

+D7 { [D0(A5)+D5(A0)][D0(A7)+D7(A0)] }
+D7 { [D1(A7)-D7(A1)][D5(A1)-D1(A5)] }
+D7 { [D2(A7)-D7(A2)][D5(A2)-D2(A5)] }
+D7 { [D3(A7)-D7(A3)][D5(A3)-D3(A5)] }
+D7 { [D4(A7)-D7(A4)][D5(A4)-D4(A5)] }
+D7 { [D6(A7)-D7(A6)][D5(A6)-D6(A5)] }

[6]
-D0 { [D0(A1)+D1(A0)][D1(A6)-D6(A1)] }
-D0 { [D0(A2)+D2(A0)][D2(A6)-D6(A2)] }
-D0 { [D0(A3)+D3(A0)][D3(A6)-D6(A3)] }
-D0 { [D0(A4)+D4(A0)][D4(A6)-D6(A4)] }
-D0 { [D0(A5)+D5(A0)][D5(A6)-D6(A5)] }
-D0 { [D0(A7)+D7(A0)][D7(A6)-D6(A7)] }

+D1 { [D0(A6)+D6(A0)][D0(A1)+D1(A0)] }
+D1 { [D2(A1)-D1(A2)][D6(A2)-D2(A6)] }
+D1 { [D3(A1)-D1(A3)][D6(A3)-D3(A6)] }
+D1 { [D4(A1)-D1(A4)][D6(A4)-D4(A6)] }
+D1 { [D5(A1)-D1(A5)][D6(A5)-D5(A6)] }
+D1 { [D7(A1)-D1(A7)][D6(A7)-D7(A6)] }

+D2 { [D0(A6)+D6(A0)][D0(A2)+D2(A0)] }
+D2 { [D1(A2)-D2(A1)][D6(A1)-D1(A6)] }
+D2 { [D3(A2)-D2(A3)][D6(A3)-D3(A6)] }

+D2 { [D4(A2)-D2(A4)][D6(A4)-D4(A6)] }
+D2 { [D5(A2)-D2(A5)][D6(A5)-D5(A6)] }
+D2 { [D7(A2)-D2(A7)][D6(A7)-D7(A6)] }

+D3 { [D0(A6)+D6(A0)][D0(A3)+D3(A0)] }
+D3 { [D1(A3)-D3(A1)][D6(A1)-D1(A6)] }
+D3 { [D2(A3)-D3(A2)][D6(A2)-D2(A6)] }
+D3 { [D4(A3)-D3(A4)][D6(A4)-D4(A6)] }
+D3 { [D5(A3)-D3(A5)][D6(A5)-D5(A6)] }
+D3 { [D7(A3)-D3(A7)][D6(A7)-D7(A6)] }

+D4 { [D0(A6)+D6(A0)][D0(A4)+D4(A0)] }
+D4 { [D1(A4)-D4(A1)][D6(A1)-D1(A6)] }
+D4 { [D2(A4)-D4(A2)][D6(A2)-D2(A6)] }
+D4 { [D3(A4)-D4(A3)][D6(A3)-D3(A6)] }
+D4 { [D5(A4)-D4(A5)][D6(A5)-D5(A6)] }
+D4 { [D7(A4)-D4(A7)][D6(A7)-D7(A6)] }

+D5 { [D0(A6)+D6(A0)][D0(A5)+D5(A0)] }
+D5 { [D1(A5)-D5(A1)][D6(A1)-D1(A6)] }
+D5 { [D2(A5)-D5(A2)][D6(A2)-D2(A6)] }
+D5 { [D3(A5)-D5(A3)][D6(A3)-D3(A6)] }
+D5 { [D4(A5)-D5(A4)][D6(A4)-D4(A6)] }
+D5 { [D7(A5)-D5(A7)][D6(A7)-D7(A6)] }

- ½ D6 { [D0(A1)+D1(A0)][D0(A1)+D1(A0)] }
- ½ D6 { [D0(A2)+D2(A0)][D0(A2)+D2(A0)] }
- ½ D6 { [D0(A3)+D3(A0)][D0(A3)+D3(A0)] }
- ½ D6 { [D0(A4)+D4(A0)][D0(A4)+D4(A0)] }
- ½ D6 { [D0(A5)+D5(A0)][D0(A5)+D5(A0)] }
+ ½ D6 { [D0(A6)+D6(A0)][D0(A6)+D6(A0)] }
- ½ D6 { [D0(A7)+D7(A0)][D0(A7)+D7(A0)] }
+ ½ D6 { [D1(A2)-D2(A1)][D1(A2)-D2(A1)] }
+ ½ D6 { [D2(A3)-D3(A2)][D2(A3)-D3(A2)] }
+ ½ D6 { [D3(A1)-D1(A3)][D3(A1)-D1(A3)] }
- ½ D6 { [D7(A6)-D6(A7)][D7(A6)-D6(A7)] }
- ½ D6 { [D6(A1)-D1(A6)][D6(A1)-D1(A6)] }
+ ½ D6 { [D1(A7)-D7(A1)][D1(A7)-D7(A1)] }
+ ½ D6 { [D5(A7)-D7(A5)][D5(A7)-D7(A5)] }
+ ½ D6 { [D7(A2)-D2(A7)][D7(A2)-D2(A7)] }
+ ½ D6 { [D2(A5)-D5(A2)][D2(A5)-D5(A2)] }
- ½ D6 { [D6(A5)-D5(A6)][D6(A5)-D5(A6)] }
+ ½ D6 { [D5(A3)-D3(A5)][D5(A3)-D3(A5)] }
- ½ D6 { [D3(A6)-D6(A3)][D3(A6)-D6(A3)] }
+ ½ D6 { [D1(A4)-D4(A1)][D1(A4)-D4(A1)] }
+ ½ D6 { [D4(A5)-D5(A4)][D4(A5)-D5(A4)] }

+ ½ D6 { [D5(A1)-D1(A5)][D5(A1)-D1(A5)] }
+ ½ D6 { [D2(A4)-D4(A2)][D2(A4)-D4(A2)] }
- ½ D6 { [D4(A6)-D6(A4)][D4(A6)-D6(A4)] }
- ½ D6 { [D6(A2)-D2(A6)][D6(A2)-D2(A6)] }
+ ½ D6 { [D3(A4)-D4(A3)][D3(A4)-D4(A3)] }
+ ½ D6 { [D4(A7)-D7(A4)][D4(A7)-D7(A4)] }
+ ½ D6 { [D7(A3)-D3(A7)][D7(A3)-D3(A7)] }

+D7 { [D0(A6)+D6(A0)][D0(A7)+D7(A0)] }
+D7 { [D1(A7)-D7(A1)][D6(A1)-D1(A6)] }
+D7 { [D2(A7)-D7(A2)][D6(A2)-D2(A6)] }
+D7 { [D3(A7)-D7(A3)][D6(A3)-D3(A6)] }
+D7 { [D4(A7)-D7(A4)][D6(A4)-D4(A6)] }
+D7 { [D5(A7)-D7(A5)][D6(A5)-D5(A6)] }

[7]
-D0 { [D0(A1)+D1(A0)][D1(A7)-D7(A1)] }
-D0 { [D0(A2)+D2(A0)][D2(A7)-D7(A2)] }
-D0 { [D0(A3)+D3(A0)][D3(A7)-D7(A3)] }
-D0 { [D0(A4)+D4(A0)][D4(A7)-D7(A4)] }
-D0 { [D0(A5)+D5(A0)][D5(A7)-D7(A5)] }
-D0 { [D0(A6)+D6(A0)][D6(A7)-D7(A6)] }

+D1 { [D0(A7)+D7(A0)][D0(A1)+D1(A0)] }
+D1 { [D2(A1)-D1(A2)][D7(A2)-D2(A7)] }
+D1 { [D3(A1)-D1(A3)][D7(A3)-D3(A7)] }
+D1 { [D4(A1)-D1(A4)][D7(A4)-D4(A7)] }
+D1 { [D5(A1)-D1(A5)][D7(A5)-D5(A7)] }
+D1 { [D6(A1)-D1(A6)][D7(A6)-D6(A7)] }

+D2 { [D0(A7)+D7(A0)][D0(A2)+D2(A0)] }
+D2 { [D1(A2)-D2(A1)][D7(A1)-D1(A7)] }
+D2 { [D3(A2)-D2(A3)][D7(A3)-D3(A7)] }
+D2 { [D4(A2)-D2(A4)][D7(A4)-D4(A7)] }
+D2 { [D5(A2)-D2(A5)][D7(A5)-D5(A7)] }
+D2 { [D6(A2)-D2(A6)][D7(A6)-D6(A7)] }

+D3 { [D0(A7)+D7(A0)][D0(A3)+D3(A0)] }
+D3 { [D1(A3)-D3(A1)][D7(A1)-D1(A7)] }
+D3 { [D2(A3)-D3(A2)][D7(A2)-D2(A7)] }
+D3 { [D4(A3)-D3(A4)][D7(A4)-D4(A7)] }
+D3 { [D5(A3)-D3(A5)][D7(A5)-D5(A7)] }
+D3 { [D6(A3)-D3(A6)][D7(A6)-D6(A7)] }

+D4 { [D0(A7)+D7(A0)][D0(A4)+D4(A0)] }
+D4 { [D1(A4)-D4(A1)][D7(A1)-D1(A7)] }
+D4 { [D2(A4)-D4(A2)][D7(A2)-D2(A7)] }
+D4 { [D3(A4)-D4(A3)][D7(A3)-D3(A7)] }
+D4 { [D5(A4)-D4(A5)][D7(A5)-D5(A7)] }
+D4 { [D6(A4)-D4(A6)][D7(A6)-D6(A7)] }

+D5 { [D0(A7)+D7(A0)][D0(A5)+D5(A0)] }
+D5 { [D1(A5)-D5(A1)][D7(A1)-D1(A7)] }
+D5 { [D2(A5)-D5(A2)][D7(A2)-D2(A7)] }
+D5 { [D3(A5)-D5(A3)][D7(A3)-D3(A7)] }
+D5 { [D4(A5)-D5(A4)][D7(A4)-D4(A7)] }
+D5 { [D6(A5)-D5(A6)][D7(A6)-D6(A7)] }

+D6 { [D0(A7)+D7(A0)][D0(A6)+D6(A0)] }
+D6 { [D1(A6)-D6(A1)][D7(A1)-D1(A7)] }
+D6 { [D2(A6)-D6(A2)][D7(A2)-D2(A7)] }
+D6 { [D3(A6)-D6(A3)][D7(A3)-D3(A7)] }
+D6 { [D4(A6)-D6(A4)][D7(A4)-D4(A7)] }
+D6 { [D5(A6)-D6(A5)][D7(A5)-D5(A7)] }

- ½ D7 { [D0(A1)+D1(A0)][D0(A1)+D1(A0)] }
- ½ D7 { [D0(A2)+D2(A0)][D0(A2)+D2(A0)] }
- ½ D7 { [D0(A3)+D3(A0)][D0(A3)+D3(A0)] }
- ½ D7 { [D0(A4)+D4(A0)][D0(A4)+D4(A0)] }
- ½ D7 { [D0(A5)+D5(A0)][D0(A5)+D5(A0)] }
- ½ D7 { [D0(A6)+D6(A0)][D0(A6)+D6(A0)] }
+ ½ D7 { [D0(A7)+D7(A0)][D0(A7)+D7(A0)] }
+ ½ D7 { [D1(A2)-D2(A1)][D1(A2)-D2(A1)] }
+ ½ D7 { [D2(A3)-D3(A2)][D2(A3)-D3(A2)] }
+ ½ D7 { [D3(A1)-D1(A3)][D3(A1)-D1(A3)] }
- ½ D7 { [D7(A6)-D6(A7)][D7(A6)-D6(A7)] }
+ ½ D7 { [D6(A1)-D1(A6)][D6(A1)-D1(A6)] }
- ½ D7 { [D1(A7)-D7(A1)][D1(A7)-D7(A1)] }
- ½ D7 { [D5(A7)-D7(A5)][D5(A7)-D7(A5)] }
- ½ D7 { [D7(A2)-D2(A7)][D7(A2)-D2(A7)] }
+ ½ D7 { [D2(A5)-D5(A2)][D2(A5)-D5(A2)] }
+ ½ D7 { [D6(A5)-D5(A6)][D6(A5)-D5(A6)] }
+ ½ D7 { [D5(A3)-D3(A5)][D5(A3)-D3(A5)] }
+ ½ D7 { [D3(A6)-D6(A3)][D3(A6)-D6(A3)] }
+ ½ D7 { [D1(A4)-D4(A1)][D1(A4)-D4(A1)] }
+ ½ D7 { [D4(A5)-D5(A4)][D4(A5)-D5(A4)] }
+ ½ D7 { [D5(A1)-D1(A5)][D5(A1)-D1(A5)] }
+ ½ D7 { [D2(A4)-D4(A2)][D2(A4)-D4(A2)] }
+ ½ D7 { [D4(A6)-D6(A4)][D4(A6)-D6(A4)] }
+ ½ D7 { [D6(A2)-D2(A6)][D6(A2)-D2(A6)] }

+ ½ D7 { [D3(A4)-D4(A3)][D3(A4)-D4(A3)] }
- ½ D7 { [D4(A7)-D7(A4)][D4(A7)-D7(A4)] }
- ½ D7 { [D7(A3)-D3(A7)][D7(A3)-D3(A7)] }

This modified Action Function can then be expressed as

Invariant(F * j) = $D_m \, T_{mn} \, u_n$

This is the divergence of the Octonion Stress-Energy-Momentum Tensor. Just as is done in Electrodynamics, both sides may be integrated over volume, here a 7-volume derived from the vector only dr. Terms on the right side where m is not zero may be converted to 6-surface integrals. When this is done, the scalar unit terms on both sides represent the Conservation of Energy, where the time rate of change in the total energy within any volume and the flux of energy leaving the volume balance the external work done on the system. The non-scalar on both sides of the equation express the Conservation of Momentum, where the time rate of change in mechanical and field momentum within the volume balance the forces impressed on the enclosing surface.

If the above process is performed on a potential function A restricted to A0, A5, A6 and A7 where all are functions of r units 0,5,6 and 7 only, the result is equivalent to the divergence of standard Electrodynamics Stress-Energy-Momentum Tensor. This is as follows.

[0]
+ ½ D0 { [D0(A5)+D5(A0)][D0(A5)+D5(A0)] }
+ ½ D0 { [D0(A6)+D6(A0)][D0(A6)+D6(A0)] }
+ ½ D0 { [D0(A7)+D7(A0)][D0(A7)+D7(A0)] }
+ ½ D0 { [D7(A6)-D6(A7)][D7(A6)-D6(A7)] }
+ ½ D0 { [D5(A7)-D7(A5)][D5(A7)-D7(A5)] }
+ ½ D0 { [D6(A5)-D5(A6)][D6(A5)-D5(A6)] }

+D5 { [D0(A6)+D6(A0)][D6(A5)-D5(A6)] }
-D5 { [D0(A7)+D7(A0)][D5(A7)-D7(A5)] }

+D6 { [D0(A7)+D7(A0)][D7(A6)-D6(A7)] }
-D6 { [D0(A5)+D5(A0)][D6(A5)-D5(A6)] }

+D7 { [D0(A5)+D5(A0)][D5(A7)-D7(A5)] }
-D7 { [D0(A6)+D6(A0)][D7(A6)-D6(A7)] }

[5]
+D0 { [D0(A7)+D7(A0)][D5(A7)-D7(A5)] }
-D0 { [D0(A6)+D6(A0)][D6(A5)-D5(A6)] }

+ ½ D5 { [D0(A5)+D5(A0)][D0(A5)+D5(A0)] }
- ½ D5 { [D0(A6)+D6(A0)][D0(A6)+D6(A0)] }
- ½ D5 { [D0(A7)+D7(A0)][D0(A7)+D7(A0)] }
+ ½ D5 { [D7(A6)-D6(A7)][D7(A6)-D6(A7)] }

- ½ D5 { [D5(A7)-D7(A5)][D5(A7)-D7(A5)] }
- ½ D5 { [D6(A5)-D5(A6)][D6(A5)-D5(A6)] }

+D6 { [D5(A7)-D7(A5)][D7(A6)-D6(A7)] }
+D6 { [D0(A5)+D5(A0)][D0(A6)+D6(A0)] }

+D7 { [D6(A5)-D5(A6)][D7(A6)-D6(A7)] }
+D7 { [D0(A5)+D5(A0)][D0(A7)+D7(A0)] }

[6]
+D0 { [D0(A5)+D5(A0)][D6(A5)-D5(A6)] }
-D0 { [D0(A7)+D7(A0)][D7(A6)-D6(A7)] }

+D5 { [D7(A6)-D6(A7)][D5(A7)-D7(A5)] }
+D5 { [D0(A6)+D6(A0)][D0(A5)+D5(A0)] }

- ½ D6 { [D0(A5)+D5(A0)][D0(A5)+D5(A0)] }
+ ½ D6 { [D0(A6)+D6(A0)][D0(A6)+D6(A0)] }
- ½ D6 { [D0(A7)+D7(A0)][D0(A7)+D7(A0)] }
- ½ D6 { [D7(A6)-D6(A7)][D7(A6)-D6(A7)] }
+ ½ D6 { [D5(A7)-D7(A5)][D5(A7)-D7(A5)] }
- ½ D6 { [D6(A5)-D5(A6)][D6(A5)-D5(A6)] }

+D7 { [D6(A5)-D5(A6)][D5(A7)-D7(A5)] }
+D7 { [D0(A6)+D6(A0)][D0(A7)+D7(A0)] }

[7]
+D0 { [D0(A6)+D6(A0)][D7(A6)-D6(A7)] }
-D0 { [D0(A5)+D5(A0)][D5(A7)-D7(A5)] }

+D5 { [D7(A6)-D6(A7)][D6(A5)-D5(A6)] }
+D5 { [D0(A7)+D7(A0)][D0(A5)+D5(A0)] }

+D6 { [D5(A7)-D7(A5)][D6(A5)-D5(A6)] }
+D6 { [D0(A7)+D7(A0)][D0(A6)+D6(A0)] }

- ½ D7 { [D0(A5)+D5(A0)][D0(A5)+D5(A0)] }
- ½ D7 { [D0(A6)+D6(A0)][D0(A6)+D6(A0)] }
+ ½ D7 { [D0(A7)+D7(A0)][D0(A7)+D7(A0)] }
- ½ D7 { [D7(A6)-D6(A7)][D7(A6)-D6(A7)] }
- ½ D7 { [D5(A7)-D7(A5)][D5(A7)-D7(A5)] }
+ ½ D7 { [D6(A5)-D5(A6)][D6(A5)-D5(A6)] }

When the differentiation outside the {} is performed on the above, the result is the expected classical Electrodynamics Action Function including the Lorentz Condition typically taken as zero.

[0]
-[-D0^2(A5)+D5^2(A5)+D6^2(A5)+D7^2(A5)][D0(A5)+D5(A0)]
-[-D0^2(A6)+D5^2(A6)+D6^2(A6)+D7^2(A6)][D0(A6)+D6(A0)]
-[-D0^2(A7)+D5^2(A7)+D6^2(A7)+D7^2(A7)][D0(A7)+D7(A0)]

+D5[D0(A0)+D5(A5)+D6(A6)+D7(A7)][D0(A5)+D5(A0)]
+D6[D0(A0)+D5(A5)+D6(A6)+D7(A7)][D0(A6)+D6(A0)]
+D7[D0(A0)+D5(A5)+D6(A6)+D7(A7)][D0(A7)+D7(A0)]

[5]
+[-D0^2(A0)+D5^2(A0)+D6^2(A0)+D7^2(A0)][D0(A5)+D5(A0)]
+[-D0^2(A6)+D5^2(A6)+D6^2(A6)+D7^2(A6)][D6(A5)-D5(A6)]
+[-D0^2(A7)+D5^2(A7)+D6^2(A7)+D7^2(A7)][D7(A5)-D5(A7)]

+D0[D0(A0)+D5(A5)+D6(A6)+D7(A7)][D0(A5)+D5(A0)]
-D6[D0(A0)+D5(A5)+D6(A6)+D7(A7)][D6(A5)-D5(A6)]
-D7[D0(A0)+D5(A5)+D6(A6)+D7(A7)][D7(A5)-D5(A7)]

[6]
+[-D0^2(A0)+D5^2(A0)+D6^2(A0)+D7^2(A0)][D0(A6)+D6(A0)]
+[-D0^2(A5)+D5^2(A5)+D6^2(A5)+D7^2(A5)][D5(A6)-D6(A5)]
+[-D0^2(A7)+D5^2(A7)+D6^2(A7)+D7^2(A7)][D7(A6)-D6(A7)]

+D0[D0(A0)+D5(A5)+D6(A6)+D7(A7)][D0(A6)+D6(A0)]
-D5[D0(A0)+D5(A5)+D6(A6)+D7(A7)][D5(A6)-D6(A5)]
-D7[D0(A0)+D5(A5)+D6(A6)+D7(A7)][D7(A6)-D6(A7)]

[7]
+[-D0^2(A0)+D5^2(A0)+D6^2(A0)+D7^2(A0)][D0(A7)+D7(A0)]
+[-D0^2(A5)+D5^2(A5)+D6^2(A5)+D7^2(A5)][D5(A7)-D7(A5)]
+[-D0^2(A6)+D5^2(A6)+D6^2(A6)+D7^2(A6)][D6(A7)-D7(A6)]

+D0[D0(A0)+D5(A5)+D6(A6)+D7(A7)][D0(A7)+D7(A0)]
-D5[D0(A0)+D5(A5)+D6(A6)+D7(A7)][D5(A7)-D7(A5)]
-D6[D0(A0)+D5(A5)+D6(A6)+D7(A7)][D6(A7)-D7(A6)]

Note that the tie-back to Electrodynamics is exclusively in units 0,5,6 and 7, and that (567) is not a valid permutation for the algebra. Hence no Quaternion sub-algebra covers Electrodynamics.

Summary and Conclusions

General Relativity, Quantum Mechanics and Electrodynamics, String Theory and this presentation in **O** should all be considered different forms of "generalized coordinate systems" that all physics students learn about in their study of classical mechanics.

It is quite improper to claim, "reality is…." Fill in your favored system. In my humble opinion, the day will never come when some brilliant mathematician/physicist will have proven to the exclusion of all other mathematical representations that nature is a particular generalized system. Mathematics is far too robust to allow this.

The validity of any generalized coordinate system is measured by what it brings to the mathematical understanding of physical reality. In the **O** system presentation above, the fields, forces, work, energy, momentum, and conservation equations from Electrodynamics are all derived from the definition of the algebra and the simple assumption that any of the possible definitions will do. Absolutely nothing is inserted; it is all given to us. I cannot imagine this is simply some cosmic coincidence.

Going to 8 dimensions, there are sufficient degrees of freedom to span more than Electrodynamics. In General Relativity, this was accomplished by allowing Gravitation induced intrinsic curvature for the 4 dimensional space-time system. This was a free choice; it was not a requirement. Within the **O** formalism above, there is an additional irrotational field that may represent Gravitation. I say, "may" because so far it only looks like it has possibilities. There is more work to be done to verify the assumption. I do not think the consideration can be done in the vacuum of not also accounting for the additional rotational field forms presented. In fact, all fields and their resultant forces are in play. The **O** formalism above provides the foundation for the dynamics of the unified system.

My considerations are currently focused on transformation properties between systems in motion. It is still a "work in progress" for me, but I will share my point of view. The diffeomorphic definition of the Ensemble Differential Form provides the transformation intrinsically, where v system is in some relative motion with respect to the r system. I do not think the formalism will care if the relative motion is constant or represents an acceleration, both will be adequately dealt with. When the relative velocity is zero, the r and v systems coincide. When the velocity moves off zero, I lean towards a functional relationship between r and v where the Jacobian is independent of the velocity. This could be accomplished by requiring the velocity effect to modify the Jacobian matrix by multiplication by some unity determinant matrix. Doing this does provide the velocity morphing between Electric and Magnetic fields. If the velocity terms are allowed themselves to be functions of time, the Ensemble Differential Form will cover accelerated frames of reference.

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